DYNAMICS OF EARTHQUAKE RESISTANT STRUCTURES

"There is nothing permanent but change."

ARISTOTLE.

DYNAMICS OF EARTHQUAKE RESISTANT STRUCTURES

FIRST EDITION

OXFORD & IBH PUBLISHING CO.
New Delhi Bombay Calcutta
1934

PREFACE

Hitherto, with a few exceptions, the design of earthquake resistant structures has been based on statical assumptions. The problem, however, is primarily a dynamical one. In the following pages, the author presents the first practical dynamical system of aseismic design.

The keynote of this monograph is brevity. The reading matter has been reduced to a minimum so that the reader may concentrate on basic principles. At the same time, most of the formulas are derived, so that as more precise seismic data become available, the new constants will be used with confidence.

It is of interest to note that certain phenomena considered paradoxes by statical standards become congruities under dynamical assumptions. For example, on page 767 of "Earthquake Damage and Earthquake Insurance," J. R. Freeman states:

> Plainly there is a somewhat paradoxical condition, difficult of precise analysis, in the undoubted facts that:
>
> 1. The earthquake motion will be far more severe on soft, mobile, unstable ground, like wet alluvial deposits or made-land, than upon rigid ground or ledge.
>
> 2. That for equal earthquake motion a carefully designed rigid building on soft yielding ground will suffer much less damage than one of equal strength located on rigid ground.

Chapter 7 shows that this "paradox" is expected theoretically.

Again, under statical methods, the maximum shears and moments in buildings occur where the height, L, is zero. It follows that the maximum earthquake damage should center at that point. This, however, is not the case. For, quoting from page 279 of T. Naito's "Earthquake Resisting Construction":

> The majority of those who have observed the different buildings in Marunouchi have drawn the conclusion that the point three-tenths of the total height is the most vulnerable spot from the standpoint of

damage by earthquake. However, a more accurate survey of the damage will indicate that the above conclusion is not always true . . .

Figures 10, 11, and 12 show that the maximum moment may occur at any point between $0.00L$ and $0.43L$, and the maximum shear between $0.00L$ and $0.75L$. Hence, the maximum earthquake damage may center at any plane of weakness between $0.00L$ and $0.75L$.

In Chap. 8 it is shown how the exterior walls of buildings can be designed to take most of the seismic shear and moment. Since walls are always used in buildings, the only added expense is in designing them to take care of the earthquake forces. In fact, large structures can be built substantially earthquake-proof at a cost rarely exceeding 10 per cent of that of an ordinary building. This outlay procures increased safety for tenants and capital investment, as well as lower insurance rates. Since earthquakes are possible in all localities, neglect to provide important buildings with aseismic strength is a questionable practice.

The author hereby expresses his sincere thanks to those who helped him to assemble, develop, or check the data set forth in the following pages, particularly to G. Back, Research Associate, U. S. Bureau of Standards; to A. Samuel, Jr., Designing Engineer, Suspension Bridges, J. A. Roebling's Sons Co., Trenton, N. J.; to A. R. Brown, Structural Engineer, Washington, D. C.; to J. M. Kerr, Chief, Structural Division, U. S. Veterans Bureau; to F. Neumann, Chief, and A. Blake, Seismology Section, U. S. Coast and Geodetic Survey, who suggested the vibrograph described on page 85; to A. J. Boase, Chief, Structural Division, Portland Cement Association, Chicago; to E. Levi, Naval Architect, Camden, N. J.; to J. Sklaroff, M. Edelman, and A. J. Creskoff, all of Philadelphia, for their invaluable assistance; and to the late J. R. Freeman, for the loan of T. Naito's "Earthquake Resisting Construction."

CONTENTS

CHAPTER 11

DYNAMICS OF EARTHQUAKE RESISTANT STRUCTURES

CHAPTER 1

M.R. Chatterjee

EARTHQUAKE HISTORY OF THE UNITED STATES

1.1. Introduction.

There is probably no part of the United States and its possessions that has not experienced at some time some degree of earthquake shock, although the sections in which earthquakes have been destructive are less extensive. The following outline indicates the extent, degree, and frequency of recorded major earthquakes within the United States and its possessions.

1.2. Eastern Region.

The first recorded major earthquake in this section occurred in the St. Lawrence Valley, in 1663. This was the site of another earthquake in 1925. Both quakes caused great landslides and were felt strongly over the northeastern quarter of the United States. New England has experienced many earthquakes, the worst being that of 1775, which shook the section from Nova Scotia to Chesapeake Bay. Maine, New Hampshire, Vermont, and Connecticut have been jolted violently from time to time. In 1884 an earthquake centering near New York was felt widely but caused little damage. Pennsylvania and West Virginia seem to have a record of minor earthquakes only. Virginia has experienced major earthquakes, the worst being the Giles County earthquake, in 1897. The Appalachian section, from North Carolina to Alabama, has suffered severely and frequently. The same is true of Georgia and Florida. The most destructive earthquake

recorded in the eastern region occurred at Charleston, S. C., in 1886, and was felt over the entire eastern half of the United States.

1.3. Mississippi Valley Region.

In 1811 this section experienced a very destructive earthquake which centered near Madrid, Mo., and was accompanied by the sinking of large parts of southeastern Missouri and northeastern Arkansas. Other major earthquakes occurred in this section in 1843 and in 1895. The region south of St. Louis, Illinois, Indiana, Ohio, and the section between South Dakota and Texas have all been shaken severely.

1.4. Rocky Mountain Region.

Earthquakes are frequent but only a few have been destructive. Severe shocks occurred in Nevada, in 1915, 1932, and 1933, and in Montana, in 1925. Utah and New Mexico have suffered severely.

1.5. Pacific Coast Region.

California is the site of considerable major earthquake activity. The Imperial Valley earthquakes, and those occurring in the Owens Valley in 1872, in San Francisco in 1906, in Santa Barbara in 1925, and at Long Beach in 1933, were all of destructive nature. The outstanding feature of the San Francisco earthquake was a 10- to 20-ft. horizontal movement of the section north of San Francisco Bay.

1.6. Alaskan Region.

Major earthquakes occur frequently. In 1899 a particularly destructive earthquake racked large sections, changed the contours of vast glacier fields, and caused a 48-ft. rise of a large area at Yakutat Bay. In 1925 a major earthquake shook the Kenai Peninsula. The Aleutian Islands suffer severely from frequent submarine earthquakes.

1.7. Hawaiian and Philippine Islands, Porto Rico, Canal Zone.

Hawaiian Islands earthquakes are numerous but rarely destructive and are almost entirely of volcanic origin. The Philippine Islands are in a region of considerable earthquake activity. The West Indies, including Porto Rico and the Virgin Islands, have suffered severely, particularly in 1868 and in 1918, the latter earthquake devastating Mayaguez. The Canal Zone experiences frequent but generally mild shocks.

1.8. Annual Earthquake Activity.

The data which follow were obtained from a report published in 1931 by N. H. Heck and R. R. Bodle of the U. S. Coast and Geodetic Survey. Although the report refers to the earthquake activity within a specific year, 1929, it is typical of any year and is given to show that there is probably no region within the United States and its possessions which is immune from the possibility of earthquake damage.

Alabama	Minor earthquake, June 13
California	Eight severe shocks. The worst were the Whittier, July 8; the Central California, Nov. 28; the Northern California, Dec. 4
Idaho	Minor quake, Oct. 1
Indiana	Moderate shock, Feb. 14
Kansas	Five quakes. Those on Sept. 20, Oct. 21, Dec. 7, severe
Louisiana	Minor shock, July 28
Maine	Four quakes. That on Feb. 5, severe
Massachusetts	Minor shock, Sept. 16
Missouri	Minor quake, Feb. 26
Montana	Three shocks. That on Feb. 15, severe
Nevada	Minor shock, Feb. 4
New Hampshire	Minor quakes on Jan. 14, Feb. 5
New York	Three shocks. That at Attica, Aug. 12, severe
Ohio	Severe quake, March 8
Oklahoma	Severe shock, Dec. 27
South Carolina	Minor shocks on Jan. 3, Oct. 27
South Dakota	Severe quake, Oct. 5
Tennessee	Minor shock, May 12

Virginia	Moderate shock, Dec. 26
Alaska	Many earthquakes. The worst were on Jan 21, Feb. 26, Mar. 6, May 26, Aug. 19
Hawaiian Islands	Many shocks. Major earthquakes on Sept. 25, Oct. 5
Porto Rico	Minor quakes
Philippine Islands	Many moderate and minor shocks
Canal Zone	Minor earthquakes

1.9. References.

DAY, A. L.: Study of Earth Movements in California, *Science*, Mar. 27, 1925.

DE BALLORE, M.: "Bibliografia general de temblores y terremotos."

FREEMAN, J. R.: "Earthquake Damage and Earthquake Insurance."

HECK, N. H.: "Earthquake Investigation in the United States."

HECK, N. H., and R. R. BODLE: "United States Earthquakes, 1929."

HOFFMAN, F. L.: "Earthquake Hazards and Insurance."

CHAPTER 2

SEISMOGRAPHY

2.1. Definitions.

According to modern theory the earth consists of a central core of molten nickel-iron 4350 miles in diameter, enveloped, in the order named, by three concentric layers of dense material totaling 1050 miles; by a mantle of highly elastic substance 700 miles thick; and, finally, by an outer shell of rock 50 miles thick, known as the *crust*. This crust is divided by innumerable planes of cleavage known as *faults* into immense irregular blocks called *fault blocks*. The surface trace of a fault plane is known as a *rift*.

Normally these fault blocks are in equilibrium, but sometimes, owing to forces acting under certain definite but as yet unknown physical laws, adjacent blocks move with respect to one another. This relative motion is called *slip*. It may be horizontal, vertical, or a combination of both. It may take place at any point below the surface to a probable maximum depth of 30 miles. Faults are known along which the blocks have slipped more than 1000 ft. A slip of such magnitude seldom takes place at one time but represents usually the accumulation of countless smaller slips.

A slip may be gradual or abrupt. If it takes place gradually, there is a similar adjustment in the contiguous strata so that, no matter how great the ultimate effect of the slip may be, it produces no immediately observable consequences. If, however, the slip is abrupt, then the surrounding strata are fractured. Thus begins a series of local movements which in turn propagate invisible, elastic earth waves. It is the surface vibration caused by these waves in passing which is popularly known as an *earthquake*.

Earthquakes caused by slip are classified as *tectonic.* Since slip presumably restores equilibrium between adjacent fault blocks, it might be reasoned that after a tectonic earthquake no further shocks need be expected. It should be kept in mind, however, that the fault remains, so that, if the forces which caused the previous slip continue to act, future earthquakes are likely.

Earthquakes due to volcanic explosions are called *volcanic.* Tectonic and volcanic earthquakes, if they occur at sea, are known as *submarine.* If a submarine earthquake is caused by vertical slip, it is usually accompanied by destructive tidal waves. The point on the earth's surface directly over the origin or *focus* of an earthquake is called the *epicenter.* The area about the epicenter, where the earthquake is usually most destructive, is known as the *epicentral zone.*

2.2. Seismographs.

Earthquakes are recorded by automatic, continuously operating instruments called *seismographs.* The fundamental idea back of the seismograph is simple and is based on certain modifications of the pendulum. A complete installation consists of two *horizontal seismometers,* set at right angles to each other, which measure the components of the seismic motion in a horizontal plane; a *vertical seismometer,* which measures the component in a vertical plane; and an automatic recording device which produces the *seismogram.* The free period of vibration of a seismograph is larger than the periods of earthquakes. If the free period of a seismograph is smaller than the seismic periods, the instrument is also known as an *accelograph.*

2.3. Invisible, Elastic Earth Waves.

Within the epicentral zone the form of the invisible, elastic earth waves is very irregular and complex. At a distant seismograph, however, they appear in more regular and simple form. The direct waves are classified as three distinct types, according to the order of arrival, the

first two types to register being known as *depth waves* and the third as *surface waves.*

2.4. Depth Waves.

The first type of the depth waves to be recorded is called *longitudinal, primary,* or *P.* It is propagated at a great depth and passes through dense material. Its displacement is parallel to the direction of propagation. The second type is known as *transverse, secondary,* or *S.* This is also propagated at great depth and passes through dense media. Its displacement, however, is perpendicular to the direction of propagation. The *P* type travels faster than the *S* type. Both have a velocity of the order of several miles per second. Any motion of an elastic solid, under no external forces, can be reproduced by combining waves of these two types.

2.5. Surface Waves.

The third type of wave recorded is a combination of *P* and *S* waves. It is also known as a *Rayleigh wave.* The velocity of the surface or Rayleigh waves is less than those of the depth waves but is also of the order of several miles per second. The amplitude of the surface waves is a maximum at the surface and a minimum at a depth of a few wave lengths. They may therefore be viewed as diverging in two dimensions only. Consequently their surface effect is stronger than that of the depth waves which diverge in three dimensions.

2.6. Fourier's Theorem.

Fourier has shown that whatever the nature of the initial motion, it can be expressed as the resultant of a number of simple harmonic motions. Again, that every wave train, of whatever form, can be expressed, in general, as the resultant of a number of simple harmonic wave trains.

2.7. Crustal Vibrations.

It has been proved mathematically that any local disturbance in an unlimited, homogeneous solid propagates

depth and surface waves and that these waves pass a point on the surface in three stages: the P waves, the S waves, and the Rayleigh or surface waves.

The vibrations of a heterogeneous solid are, of course, more complicated than those of a homogeneous solid. A complete theory regarding the vibrations is not yet available. Certain essential points, however, may be deduced logically.

Thus, an impulse at the origin will propagate P and S waves. These will diverge with velocities appropriate to their types. Every time they enter matter with different physical properties, they will undergo modification. In part they will be reflected and in part they will be refracted, with changes in velocities and directions. Every point of the boundary between two media will act as a new origin, and the waves will first affect a given point in the second medium after an interval equal to the shortest time of transmission of a wave of the given type from the origin to the point.

Proceeding to the case of the earth's crust where the physical properties vary from point to point, the same general results will evidently hold. Thus, simultaneously with the slip, P and S waves are propagated and depart with characteristic velocities. Internal reflection forces some waves to travel by more circuitous routes than the main waves, and such reflected waves will arrive later. Sudden displacements of a particle occur as the waves pass it. Where there is a definite boundary between materials, definite reflected waves will be produced. These will be recorded individually. Where the transition is continuous, however, the vibrations will continue over a long interval and will not yield definite shocks.

Love has shown that the velocity of the surface or Rayleigh waves depends on the period of vibration. Since it is possible to represent the initial disturbance as a combination of waves with a wide range of periods, it follows that surface waves of different periods will diverge at various velocities and pass a given station at different times. Thus the motion at a distant station due to the

surface waves should consist of a long series of vibrations of varying periods.

Observation shows that the actual phenomena are in close accord with what is expected theoretically. It is found that the first effect of an earthquake at a distant station is a sudden impulse away from the focus. This corresponds to the arrival of the *P* waves. It is followed by another jolt which signals the arrival of the *S* waves. These are preceded and followed by trains of waves which are identified as the reflected and surface waves.

2.8. Visible Earth Waves.

In addition to the invisible, elastic earth waves, short surface waves are sometimes seen during a major earthquake. They are thought to be caused by the passage of depth waves from highly elastic to poorly elastic media and appear in the soft ground in the epicentral zone. Their velocity is of the order of several feet per second.

2.9. Sound Waves.

Major earthquakes are often preceded or accompanied by dull hums or low-pitched roars. These are believed to be due to the vertical vibration of the ground.

2.10. Duration of Earthquakes.

Destructive earthquakes may last for several minutes. They may cease and recur. The most destructive part of a major earthquake, however, occurs early and lasts only a short time, so that the fate of any structure is usually decided within the first minute from the beginning.

2.11. Description of a Major Earthquake.

On Sept. 1, 1923, Southeastern Japan was devastated by an earthquake which with the ensuing fire caused 141,720 deaths and the destruction of 580,397 buildings. Dr. Imamura, director of the Tokyo Seismological Observatory, describes the earthquake as follows:

I was sitting in the Seismological Institute.

At first the movement was slow and feeble, so that I did not take it to be the forerunner of so big a shock. As usual, I began to estimate the duration of the tremors and determined, if possible, to ascertain the directions of the principal movements.

Soon the vibration became large, and after three or four seconds from the commencement, I felt the shock very strong indeed. Seven or eight seconds passed and the building was shaking to an extraordinary extent, but I considered these movements not yet to be the principal portion.

When I counted the *twelfth second* from the start, there arrived a very big vibration, which I took at once as the beginning of the principal portion. Now the motion, instead of becoming less and less as usual, went on to increase its intensity very quickly and after *four* or *five seconds* I felt it to have reached its strongest. During this epoch the tiles were showering down from the roof making a loud noise and I wondered whether the building could stand or not. I realized the directions of the principal movements distinctly and found them to have been about Northwest and Southeast.

During the following ten seconds the motion, although still violent, became somewhat less severe, and its character was gradually changing into slower but bigger vibrations. For the next few minutes I felt an undulation like that which we experience on a boat in windy weather, being now and then threatened by severe after shocks. Five minutes from the beginning I stood up and went over to see the instruments.

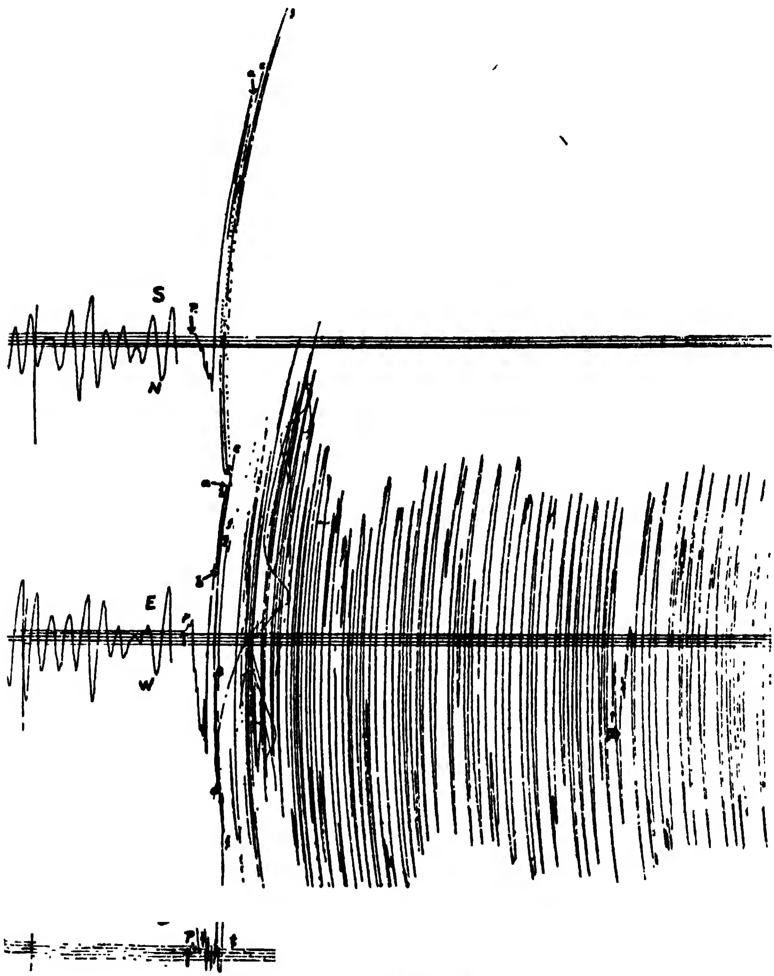
Only the east-west seismometer of one seismograph had functioned throughout. Its seismogram, shown in Fig. 1, furnishes the only complete data concerning the characteristics of the Tokyo earthquake in the epicentral zone.

According to Dr. Imamura's description of the earthquake, the period during which the maximum force was exerted was between the *twelfth* and *seventeenth* seconds from the beginning. Figure 2 shows the actual motion of an earth particle at the site of the Seismological Institute during that interval.

2.12. Earth Tilt as a Harbinger of Earthquakes.

There is a good deal of evidence to the effect that there is an intimate connection between *earth tilt, i.e.*, progressive changes in the elevation of the earth's surface in any

locality, and the coming of an earthquake. Earth tilt may be determined by successive lines of precise levels, or by a sensitive recording clinometer such as the *Ishimoto tiltgraph.*



Time in Minutes

Fig. 1.—Seismogram of Tokyo quake, 1923.

Earth tilt, however, cannot be considered as an infallible sign of an impending earthquake as it may also be caused by meteorological conditions.

2.13. References.

DAVISON, C.: "Manual of Seismology."

FREEMAN, J. R.: "Earthquake Damage and Earthquake Insurance."

HECK, N. H., and F. NEUMANN: Long Beach Seismogram, *Eng. News-Record*, June 22, 1933.

IMAMURA, A.: *Seismological Notes*, No. 6, Imperial Earthquake Investigation Committee.

———: Topographical Changes Accompanying Earthquakes or Volcanic Eruptions, *Publication* 25, Imperial Earthquake Investigation Committee.

JEFFREYS, H.: "The Earth."

KNOTT, C. G.: "The Physics of Earthquake Phenomena."

LAMB, H.: "The Propagation of Tremors over the Surface of an Elastic Solid."

LOVE, A. E. H.: "A Treatise on the Mathematical Theory of Elasticity."

MACELWANE, J. B.: The Interior of the Earth, *Bull. Seis. Soc. Amer.*, June, 1924.

McCOMB, H. E.: Tilt Compensation Seismometer, *Bull. Seis. Soc. Amer.*, March, 1931.

MILNE, J.: "Seismology."

NEUMANN, F.: Principles Underlying the Interpretation of Seismograms, *U. S. Coast and Geodetic Survey.*

REID, H. F.: Report of the State Earthquake Investigation Commission of California, vol. 2.

ROGERS, F. J.: Experiments with a Shaking Machine, *Bull. Seis. Soc. Amer.*, September, 1930.

SIEBERG, A.: "Handbuch der Erdbebenkunde."

SUYEHIRO, K.: Engineering Seismology, *Proc. Amer. Soc. Civil Eng.*, May, 1932.

WALKER, G. W.: "Modern Seismology."

WENNER, F.: Development of Seismological Instruments at the Bureau of Standards, *Bull. Seis. Soc. Amer.*, March, 1932.

WOOD, H. O.: *Proc. Amer. Soc. Civil Eng.*, November, 1928.

CHAPTER 3

SEISMIC MOTION OF A PARTICLE

3.1. Introduction.

The term *particle* as used here, may be a body of any size or shape, provided that all the forces acting on it are regarded as applied at a single point and that the motion of the point determines the motion of the body.

When an elastic particle, such as any part of the earth's crust, is acted on by *impact*, *i.e.*, the sudden application or release of a force, it is thrown into *free vibration*. Every locality, if acted on by impact, will vibrate in a mode peculiar to itself.

If, however, the particle is acted on by a *periodic force*, *i.e.*, a force which varies with time, the ensuing vibration will not be characteristic of the particle or the locality but will assume eventually the mode of the periodic force. This is called *forced vibration*. If the period of the periodic force approaches the value of the free period of the locality, the phenomenon known as *synchronism* occurs during which the region experiences a sudden and abnormal increase of motion.

3.2. Seismic Motion of a Particle.

The amplitude of a particle during an earthquake depends on:

a. The characteristics of the seismic waves.
b. The nature of the strata.
c. The depth of the particle below the surface.

3.3. Effects of the Characteristics of the Seismic Waves.

The characteristics of the invisible, elastic earth waves which affect the motion of a particle are:

a. The *amplitude* of vibration, defined as the maximum displacement of a wave crest either parallel or perpendicular to the axis of propagation.

b. The *period* of vibration, defined as the time interval between consecutive wave crests.

c. The *number of successive similar vibrations* in the same direction.

3.4. Effect of Wave Amplitude and Period.

Theoretically, the motion imparted to a particle by an invisible, elastic earth wave will be directly proportional to the wave amplitude and inversely proportional to the square of its period.

The consensus of scientific opinion seems to be that destructive earthquakes have associated with them a definite bracket of periods ranging between 1.0 and 1.5 sec. This is known as the *dangerous bracket of periods.* Accuracy of the conclusion referred to above is, however, open to question. As Blake points out, there seems to be no real reason why the dangerous bracket should not include all earthquake periods to which buildings may respond.

3.5. Effect of Successive Similar Vibrations.

Irregular and discontinuous waves impart similar motion, but when the successive waves are of equal period, continuous, and in the same direction, then the motion of the particle becomes cumulative and excessive. This is similar to what happens when a swing is pushed at equal intervals, with equal force, and in the same direction; it passes through increasingly longer arcs.

It seems to be a matter of chance as to whether successive vibrations will be similar, with the probabilities in favor of dissimilarity. Figure 2, however, shows that successive waves were practically similar during the most destructive part of the Tokyo earthquake.

3.6. Effect of the Nature of the Strata.

It has been demonstrated that the period of vibration of the surface layer of a stratified crust depends on the

elasticity of the layer and on its thickness. If the layer is highly elastic and deep, it will vibrate in a characteristic period. If, however, it possesses little elasticity, say that it is marsh or semifluid ground, then there is likelihood of

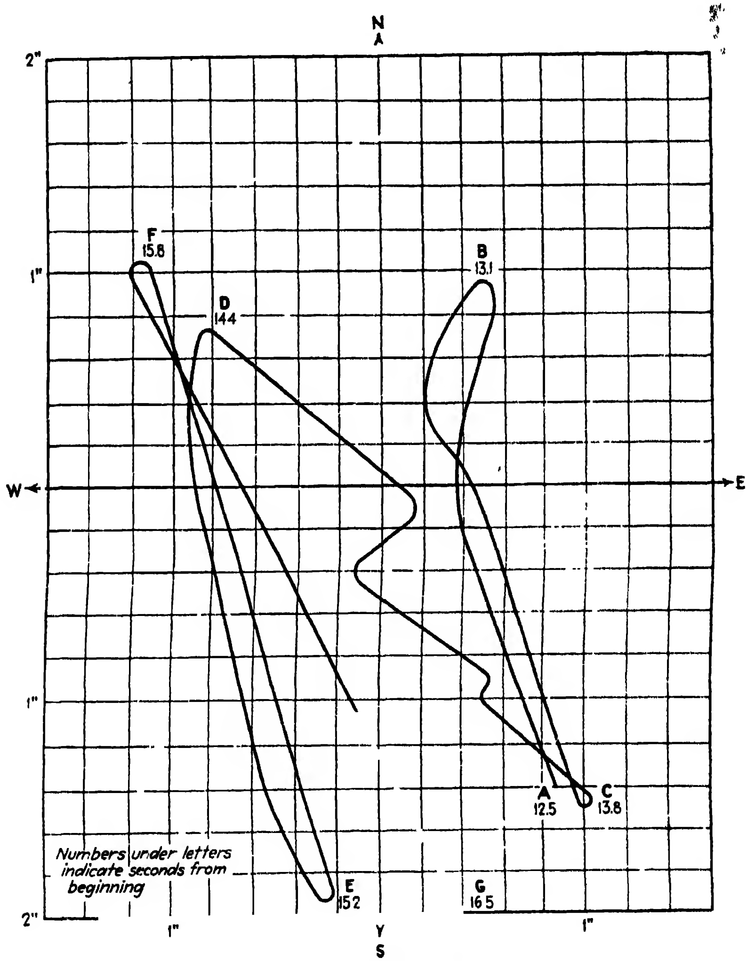


FIG. 2.—Horizontal motion of particle during Tokyo quake, 1923.

vibration in several dominant periods. Consequently, the chances of synchronism occurring during an earthquake are greater. It is a matter of observation that semifluid ground is more responsive to synchronism than more

Suyehiro devised an instrument called the *Suyehiro vibration analyzer* which utilizes the principle of selective synchronism to detect the dominant periods of a locality. The instrument consists of 13 compound pendulums adjusted to free periods varying from 0.2 to 1.8 sec., each pendulum being connected by magnifying linkage to a stylus which records on a revolving drum.

Other factors being equal, the seismic motion of a particle will depend on the nature of the local strata. Thus, a particle in bedrock will move least. One in dry, cohesive soil will be displaced more. Finally, a particle in marsh, or soft loose fill, will have the greatest amplitude. The motion of a particle in wet soil will be greater than that of one in a similar but dry soil. The greater motion of a soft layer is not prevented by an overlying thin stratum of drier and more compact material, a particle in the overlying stratum being displaced in the same general order as one in the soft layer below.

3.7. Effect of the Depth of the Particle below the Surface.

It is a common experience for mine workers to be unaware that an earthquake is in progress on the surface. The following experience, related by J. N. Wynne, appeared in *Mining and Metallurgy*, January, 1932.

> The quake occurred on a mine holiday and only some dozen men were underground at the time, engaged in concreting a pump chamber. By the time that I was able to get across to the mine, these men had been brought to the surface and were astonished to hear of the quake; they were absolutely ignorant of any quake at all; yet on the surface a generating plant had been thrown from its loadings; an 80-ft. smokestack, which I had seen waving about, was split from bottom to top. Less than a quarter of a mile away, thousands of tons of andesite had been hurled from the cliff faces. On later inspection I found that absolutely no damage had been sustained by the underground workings, although there were large areas of open stopes above.

Experiments by Omori and Sekiya indicated that a particle located at some depth is displaced less than one near the surface. Suyehiro and Nasu conducted a series of experiments to investigate the ratio between the ampli-

tude of a particle on the surface of the Tanna Basin and that of one located in a tunnel 524 ft. below. The results may be expressed as,

$$Y = e^{-\frac{K}{T_p}} \qquad (1)$$

where Y = ratio between subsurface and surface amplitudes.
e = Napierian base = 2.71828.
K = a constant. Experimental and mathematical considerations suggest that K is proportional to the depth of the particle below the surface.
T_p = period of vibration of the earthquake.

Figure 3 shows the effect of depth on amplitude. It is evident that the amplitude imparted to a particle decreases as its depth increases. The curve in Fig. 3 is based on an average value of the periods of destructive earthquakes.

3.8. Orbit of a Particle.

During an earthquake a particle vibrates in all three dimensions. Its orbit is therefore very complex. As a matter of fact, seismographic records reveal that the path of a particle is complicated even if motion in only one plane is considered. In the majority of cases the projection of the space orbit upon an horizontal or vertical plane discloses a crude, circular or elliptical path, very irregular in outline and with many abrupt changes in direction.

3.9. Extent of Horizontal Motion.

Excluding the case of synchronism, the maximum amplitude of a surface particle during a destructive earthquake is seldom greater than

0.25 in. for a particle in bedrock.
1.00 in. for a particle in dry, cohesive soil.
4.00 in. for a particle in marsh or soft, wet fill.

Earthquakes causing greater amplitudes have occurred, but their periods were large and the relatively languid vibrations caused little damage.

Figure 2 shows the horizontal motion of a surface particle during the most destructive part of the Tokyo earthquake. The particle was in well-consolidated fill. The average period of vibration of the earthquake during this interval was 1.3 sec. The maximum amplitude of the particle was 1.9 in.

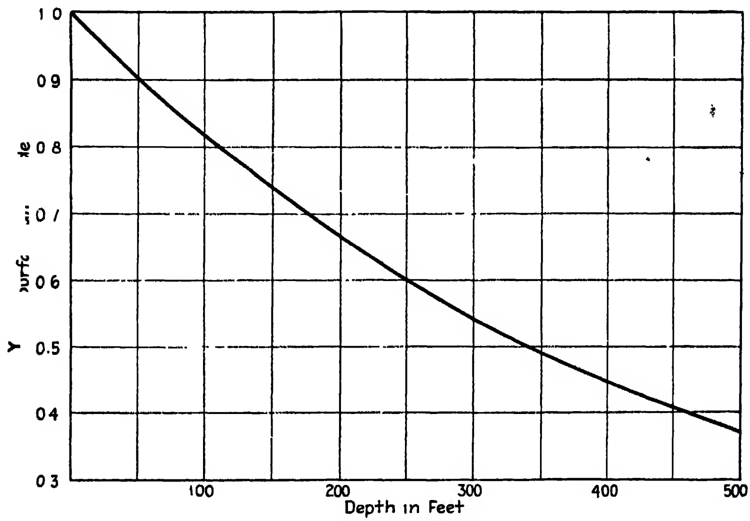


FIG. 3.—Effect of depth on amplitude.

3.10. Extent of Vertical Motion.

The vertical amplitude of a particle in the epicentral zone is probably of the order of the horizontal amplitude. An idea of the large vertical amplitude possible is obtained from the testimony of a Japanese schoolmaster in the epicentral zone during the Tajima earthquake, in 1925. He stated that, during the earthquake, coins saved by his pupils in a closed tin can were seen to cast off the lid and hop out. The can itself, although it wobbled about, remained upright.

3.11. Velocity of a Particle.

The seismic waves travel at high velocity of the order of several miles per second. Any particle on which they

act, however, is displaced in the order of a few inches per second. Particles therefore travel at low velocity. The relation between seismic wave velocity and particle velocity may be illustrated by tossing a pebble into a pond on the surface of which a toy boat is floating. It will be noticed that the water waves will spread out rapidly and cover considerable ground, whereas the toy boat is displaced but slightly and moves slowly.

3.12. References.

BLAKE, A.: The Recording of Strong Seismic Motion, *Bull. Seis. Soc. Amer.*, July, 1933.

FREEMAN, J. R.: "Earthquake Damage and Earthquake Insurance."

JACOBSEN, L. S.: Motion of a Soil Subjected to a Simple Harmonic Ground Vibration, *Bull. Seis. Soc. Amer.*, September, 1930.

MILNE, J.: Effect of Depth on Amplitude, *Trans. Seis. Soc. Japan*, vol. 10.

OMORI, F., and S. SEKIYA: Effect of Depth on Amplitude, *Jour. Coll. Sci., Imp. Univ. Tokyo*, vol. 4.

ROGERS, F. J.: Experiments with a Shaking Table, *Bull. Seis. Soc. Amer.*, September, 1930.

SUYEHIRO, K.: Engineering Seismology, *Proc. Amer. Soc. Civil Eng.*, May, 1932.

CHAPTER 4

DYNAMICS OF EARTHQUAKE MOTION

4.1. Simple Harmonic Motion.

If the uniform motion of a particle in a circular path be projected along a diameter, the vibration of the projected point to and fro along the diameter constitutes *simple harmonic motion.* This may be represented graphically by a cosine curve, time being the abscissa and displacement the ordinate.

Fourier has demonstrated that the most complex vibration of a particle may, in so far as friction can be neglected, be expressed as the resultant of a series of simple harmonic vibrations. Each of the simple harmonic vibrations is called a component of the complex vibration.

4.2. Displacement, Velocity, Acceleration.

If particle P in Fig. 4 moves around the circumference at a constant angular velocity n, the vibration of the projection of P, namely, P_x, on the diameter AB, constitutes simple harmonic motion. It is evident that, for every complete revolution of P, P_x vibrates along the diameter from A to B and back again to A.

Assume that P is at A when the time t equals zero. Then, when time equals t, the angle AOP traversed by the radius vector of P will equal nt. The distance of P_x from the central position O will then be

$$x = r \cos nt \tag{2}$$

where x is the displacement at time, t, and r is the radius.

The angle nt is called the *phase.* The maximum displacement of P_x, namely, r, is known as the *amplitude.* The time it takes P to move completely around the circumfer-

ence, or P_x to vibrate along the diameter from A to B and back again to A, is called the *period* T. Obviously,

$$T = \frac{2\pi}{} \tag{3}$$

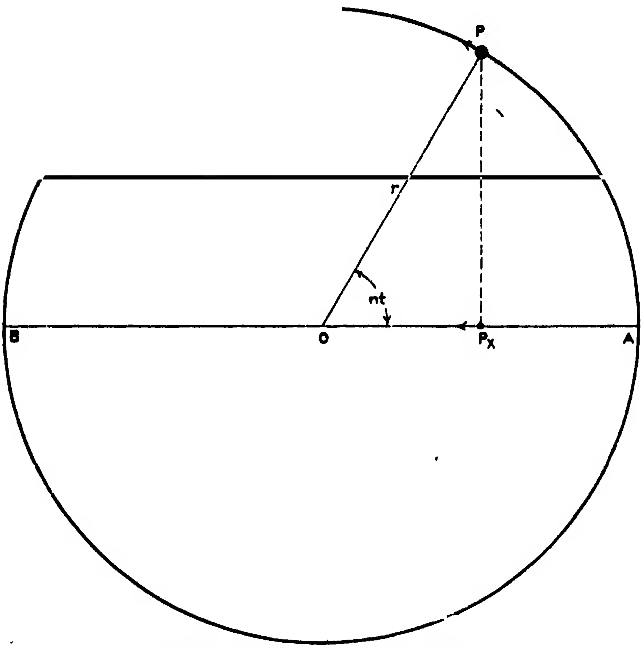


FIG. 4.—Simple harmonic motion.

The number of revolutions per unit of time is known as the *frequency* f' and equals the reciprocal of the period, or

$$f' = \frac{n}{2\pi} \tag{4}$$

Equation (2) shows that the displacement is a maximum when $nt = 0$, π, etc.; and a minimum when $nt = \pi/2$, $3\pi/2$, etc.

The *velocity* v of P_x in any position is given by the first derivative of x with respect to t, or

$$v = \frac{dx}{dt} = -nr \sin nt \tag{5}$$

Equation (5) shows that the velocity is a maximum when $nt = \pi/2, 3\pi/2$, etc.; and a minimum when $nt = 0, \pi$, etc.

The *acceleration* a of P_x in any position is given by the second derivative of x with respect to t, or

$$a = \frac{d^2x}{dt^2} = -n^2 r \cos nt \tag{6}$$

Equation (6) shows that the acceleration is a maximum when $nt = 0, \pi$, etc.; and a minimum when $nt = \pi/2, 3\pi/2$, etc.

4.3. Free Vibrations without Friction.

By Eq. (2), $x = r \cos nt$. Substituting x for $(r \cos nt)$ in Eq. (6),

$$\frac{d^2x}{dt^2} = -n^2 x \tag{7}$$

and

$$\frac{d^2x}{dt^2} + n^2 x = 0 \tag{8}$$

Equation (7) shows that in simple harmonic motion, the acceleration is directly proportional to the displacement and directionally opposed to it. Equation (8) covers all cases of the components of the motion of a dynamical system vibrating through a small range about a center of equilibrium and in free, frictionless, rectilinear fashion.

Equation (8) may also be expressed as

$$\frac{d^2x}{dt^2} + \frac{K}{M}x = 0 \tag{9}$$

where K is the coefficient of elasticity, and M is the coefficient of mass.

Comparing Eqs. (8) and (9), it is obvious that

$$n^2 = \frac{K}{M} \text{ and } n = \sqrt{\frac{K}{M}} \tag{10}$$

Substituting $(K/M)^{1/2}$ for n, Eq. (3) becomes

$$T = 2\pi\sqrt{\frac{M}{K}} \tag{11}$$

The period of vibration, T, is therefore directly proportional to the square root of the mass of the particle and inversely proportional to the square root of its elasticity.

4.4. Free vs. Forced Vibrations.

When an elastic particle is acted on by impact, it is thrown into free vibration. Theoretically, an elastic body can vibrate freely in an infinite number of ways or *modes*. There are certain simple modes, however, in each of which the particle executes simple harmonic motion. The mode of largest period is known as the *fundamental*, and those of lesser periods are called *harmonics*. The fundamental and harmonics are known as the *normal modes*. Any free motion is a combination of normal modes.

If, instead of impact, an impressed periodic force acts on the particle, the ensuing vibration is called *forced*. Two distinctions between free and forced vibrations are: the period of a free vibration depends on the constitution of the particle, whereas that of a forced vibration is determined by the periodic force. Again, a free vibration dies out sooner or later because of friction, but a forced vibration continues as long as the periodic force acts.

To illustrate: Suppose that a periodic force is impressed on the particle. At first an important free vibration occurs. This becomes insignificant as the periodic force prevails. The particle is now vibrating in the period of the periodic force. This condition continues until the periodic force is removed. The forced vibration then changes into a free vibration with a corresponding change in period.

The initial free vibration is important for the reason that, should the free and forced periods be equal, an abrupt and abnormal increase in amplitude will occur. This is what is called *synchronism.*

4.5. Forced Vibrations without Friction.

Let the impressed simple harmonic vibration x be defined by

$$x = F \cos pt \tag{12}$$

where F = amplitude.
p = forced angular velocity.
t = time.

The differential equation of this forced vibration is

$$\frac{d^2x}{dt^2} + \frac{K}{M}x = \frac{F}{M} \cos pt \tag{13}$$

If f be substituted for F/M, and n^2 from Eq. (10) for K/M, Eq. (13) becomes

$$\frac{d^2x}{dt^2} + n^2x = f \cos pt \tag{14}$$

the solution of which is

$$x = A \cos nt + B \sin nt + \frac{f \cos pt}{n^2 - p^2} \tag{15}$$

The first two terms on the right-hand side of Eq. (15) represent the free vibration whose period is $T = 2\pi/n$. The arbitrary constants A and B are determined by the initial conditions and may have any values. The last term on the right-hand side represents the forced vibration whose period is $T_p = 2\pi/p$. When the value of the forced period approaches that of the free period, this last term becomes very large. This explains why abnormal amplitudes occur during synchronism.

4.6. Free Vibrations with Friction.

The conception of motion without friction is, of course, an ideal one. Actually, frictional forces are always present; in fact, it is due to their influence that free vibrations gradually cease. These frictional or damping forces are the result of internal and external conditions. For example, they may be due to the internal viscosity or to external sliding friction. The frictional forces convert the energy of vibration into heat energy and constantly diminish the amplitude of vibration.

Experimental results show that internal frictional resistance varies as the velocity. If the velocity is low, as it is

in the case of the seismic motion of a particle, the internal frictional resistance is approximately proportional to the first power of the velocity.

To represent the case of free vibrations with friction, a term corresponding to a force proportional to the first power of the velocity is therefore added to Eq. (9), which becomes

$$\frac{d^2x}{dt^2} + \frac{K}{M}x + \frac{R}{M}\frac{dx}{dt} = 0 \tag{16}$$

where R is the coefficient of frictional resistance. Substituting n^2 for K/M, and k for R/M, Eq. (16) becomes

$$\frac{d^2x}{dt^2} + n^2x + k\frac{dx}{dt} = 0 \tag{17}$$

Let $x = ye^{-\frac{1}{2}kt}$. Then, by substitution, Eq. (17) becomes

$$\frac{d^2y}{dt^2} + \left(n^2 - \frac{1}{4}k^2\right)y = 0 \tag{18}$$

4.7. Free Vibrations—Friction Small.

If the friction in the system is relatively small, *i.e.*, k is less than $2n$, the solution of Eq. (18) becomes, if n'^2 be substituted for $(n^2 - \frac{1}{4}k^2)$,

$$y = A \cos n't + B \sin n't \tag{19}$$

so that

$$x = e^{-\frac{1}{2}kt}(A \cos n't + B \sin n't) \tag{20}$$

By a suitable choice of r and θ, $A = r \cos \theta$ and $B = -r \sin \theta$. Then, by substitution and reduction, Eq. (20) becomes

$$x = re^{-\frac{1}{2}kt} \cos (n't + \theta) \tag{21}$$

where $re^{-\frac{1}{2}kt}$ = amplitude.
x = displacement.
e = Napierian base.
θ = initial phase.
$(n't + \theta)$ = total phase.
T' = period of vibration = $2\pi/n'$.

Equation (21) shows that the amplitude sinks asymptotically to zero as the time t increases. Also, the smaller

k is, the smaller are the differences $(n' - n)$ and $(T - T')$. Consequently, if the friction in the system is small, its effect on the period is negligible.

4.8. Free Vibrations—Friction Large.

If the friction in the system is relatively large, *i.e.*, k is greater than $2n$, the solution of Eq. (18) becomes

$$y = Ae^{\sqrt{(\frac{1}{4}k^2 - n^2)}t} + Be^{-\sqrt{(\frac{1}{4}k^2 - n^2)}t} \tag{22}$$

Let

$$U = \tfrac{1}{2}k - (\tfrac{1}{4}k^2 - n^2)^{\frac{1}{2}}$$

and

$$V = \tfrac{1}{2}k + (\tfrac{1}{4}k^2 - n^2)^{\frac{1}{2}}$$

Then, by substitution, $(x = ye^{-\frac{1}{2}kt})$ becomes

$$x = Ae^{-Ut} + Be^{-Vt} \tag{23}$$

Equation (23) shows that the particle does not vibrate but settles down gradually to its zero or central position.

4.9. Forced Vibrations with Friction.

To represent the case of forced vibrations with friction, a term corresponding to a force proportional to the first power of the velocity is added to differential equation (13), which becomes

$$\frac{d^2x}{dt^2} + \frac{K}{M}x + \frac{R}{M}\frac{dx}{dt} = \frac{F}{M}\cos pt \tag{24}$$

Substituting n^2 for K/M, k for R/M, and f for F/M,

$$\frac{d^2x}{dt^2} + n^2x + k\frac{dx}{dt} = f\cos pt \tag{25}$$

The particular solution of Eq. (25) is

$$x = \frac{f}{Z}\cos(pt - \phi) \tag{26}$$

where

$$Z = \sqrt{(n^2 - p^2)^2 + k^2p^2} \tag{27}$$

and

$$\tan \phi = \frac{kp}{n^2 - p^2} \tag{28}$$

Z is positive, and ϕ may be assumed to lie between 0 and π.

Equation (26) contains a term which represents the relation between the elasticity and the inertia and another which represents the energy required to compensate for the energy losses.

General equation (25) is still satisfied if terms representing a free vibration are added to Eq. (26). These terms are required if a solution capable of adjustment to the arbitrary initial conditions is desired. The forced vibration given by Eq. (26) holds as soon as the free vibration ceases and as long as the impressed force continues to act.

4.10. Effect of Frictional Resistance on the Amplitude during Synchronism.

As the ratio between the free and forced periods diverges from unity, the amplitude decreases rapidly, the rate of decrease being governed by the amount of frictional resistance in the system.

Thus, for small frictional resistances, *i.e.*, where k is less than $2n$, the smaller the resistance, the greater the maximum amplitude; the more rapid the descent from that maximum as the ratio of the periods departs from unity; and the narrower the range over which synchronism is felt.

For large frictional resistances, *i.e.*, where k is greater than $2n$, the larger the resistance, the smaller the maximum amplitude; the slower the descent from that maximum as the ratio between the periods departs from unity; and the wider the range over which synchronism is felt.

4.11. Generalizations.

The particle discussed in Par. 4.2 to 4.9 executed motion in one dimension only and is said to have only *one degree* of freedom. Suppose, however, that an elastic body has *m degrees* of freedom, *i.e.*, it can perform translation and

rotation with respect to the axes x, y, and z. For its particles there are, in general, m possible free modes in each of which a particle vibrates as if it had only one degree of freedom. Bernoulli shows that the most general free motion of an elastic body about a center of stable equilibrium may be regarded as made up of m free modes, each with its own amplitude and initial phase.

The study of the tridimensional vibrations of an elastic body therefore resolves itself into a study of its vibrations in each dimension. Rayleigh shows that even when it is difficult to ascertain the precise character of a particular vibration, a close approximate solution can always be obtained by assuming a type which can be judged on independent grounds as being a fair representation of the unknown vibration.

Thus, if a particle acts as if it is approximating simple harmonic motion (see Fig. 3), there can be no great error involved in assuming that it is vibrating in simple harmonic fashion.

4.12. References.

BIOT, M.: Theory of Elastic Systems Vibrating under Transient Impuls. with an Application to Earthquake-proof Buildings, *Proc. Nat. Acad. Sci.*, February, 1933.

FREEMAN, J. R.: "Earthquake Damage and Earthquake Insurance."

JACOBSEN, L. S.: Steady Forced Vibration of a System of One Degree of Freedom under the Influence of Various Types of Damping, *Bull. Seis. Soc. Amer.*, September, 1930.

LAMB, H.: "Dynamical Theory of Sound."

PRESCOTT, J.: "Applied Elasticity."

RAYLEIGH, LORD: "Theory of Sound," vol. 1.

SUYEHIRO, K.: Engineering Seismology, *Proc. Amer. Soc. Civil Eng.*, May, 1932

TIMOSHENKO, S.: "Applied Elasticity."

WOOD, A. B.: "Text Book of Sound."

CHAPTER 5

DEFLECTIONS OF BEAMS IN MOMENT AND SHEAR

If tensile, compressive, and shearing stresses exist in an element of a beam, it is an experimental fact fundamental to the subject of elasticity that each stress produces its own strain and deflection just as if no other stress were present, the total deflection of the beam at any point being equal to the sum of the individual deflections due to the moment (tension and compression) and shear at the point.

5.1. Deflection Due to Moment.

For deflections within the elastic limit, the conventional theory of homogeneous, isotropic beams shows that

$$EIy = \int\int M dx dx \tag{29}$$

where E = Young's modulus.
I = moment of inertia of the section.
y = deflection at any point x.
M = moment at x.

The first derivative of y with respect to x gives the slope $dy/dx = \tan\theta$ at any point, or

$$EI\frac{dy}{dx} = \int M dx \tag{30}$$

where θ is the angle which the tangent to the deflection curve at x makes with the undeflected axis of the beam.

The second derivative of y with respect to x gives the moment M at any point, or

$$EI\frac{d^2y}{dx^2} = M \tag{31}$$

The third derivative of y with respect to x gives the total shear V at any point, or

$$EI\frac{d^3y}{dx^3} = V \tag{32}$$

The fourth derivative of y with respect to x gives the mass-acceleration force F at any point, or

$$EI\frac{d^4y}{dx^4} = -F \tag{33}$$

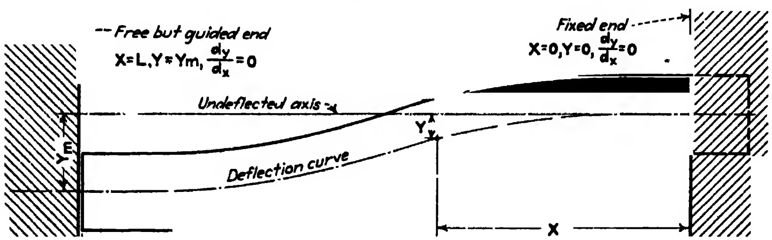


FIG. 5.—Moment deflection of fixed-free but guided beam.

F being directionally opposed to V.

But, according to Newton,

$$F = ma = \frac{w}{g}a$$

where w = weight per unit length.
m = mass per unit length.
g = acceleration due to gravity.
a = acceleration at any point x and equals d^2y/dt^2.

Therefore,

$$F = \frac{w}{g}\frac{d^2y}{dt^2} \tag{34}$$

and, by substitution, Eq. (33) becomes

$$\frac{d^4y}{dx^4} = -\frac{w}{EIg}\frac{d^2y}{dt^2} \tag{35}$$

5.2. Moment Deflection of a Fixed-free but Guided Beam.

The term *fixed-free but guided,* as used here, defines the end conditions of a beam which is fixed for translation and

rotation where $x = 0$, and free for translation but fixed for rotation where $x = L$. Figure 5 shows such a beam with a concentrated static load P acting at $x = L$.

The vertical shear V is obtained by substituting P for V in Eq. (32). Therefore,

$$EI\frac{d^3y}{dx^3} = -P \tag{36}$$

Integrating Eq. (36) to get the moment M,

$$EI\frac{d^2y}{dx^2} = -Px + A \tag{37}$$

Integrating Eq. (37) to get the slope dy/dx,

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + Ax + B \tag{38}$$

Integrating Eq. (38) to get the deflection y,

$$EIy = -\frac{Px^3}{6} + \frac{Ax^2}{2} + Bx + C \tag{39}$$

At the fixed end, $x = 0$, $y = 0$, $dy/dx = 0$. Substituting these values of x, y, and dy/dx into Eqs. (39) and (38), it is found that, $B = 0$, $C = 0$. Equations (38) and (39) therefore become

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + Ax \tag{40}$$

$$EIy = -\frac{Px^3}{6} + \frac{Ax^2}{2} \tag{41}$$

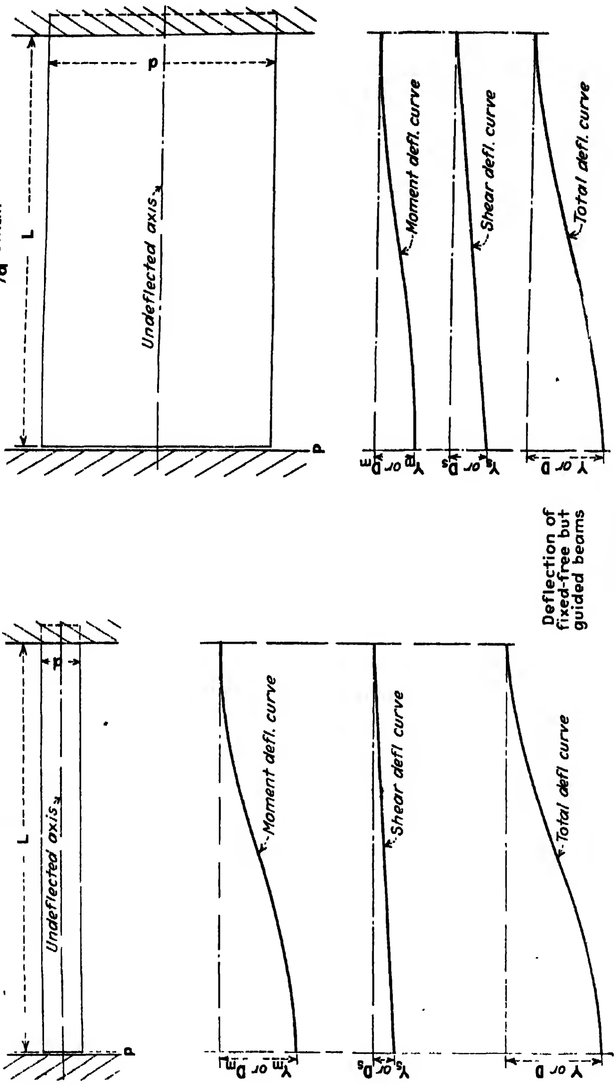
At the free-but-guided end, $x = L$, $y = y_m$, $dy/dx = 0$.

Substituting these values of x, y, and dy/dx into Eq. (40), and solving for A,

$$A = \frac{PL}{2} \tag{42}$$

Substituting $PL/2$ for A in Eq. (41),

$$EIy = \frac{Px^2}{12}(3L - 2x) \tag{43}$$



Deflection of fixed-free but guided beams

FIG. 5a.

and,

$$y_m = \frac{PL^3}{12EI} \tag{44}$$

where y_m is the maximum deflection due to moment.

5.3. Shear Strains.

It is possible to regard shear strains as due to the progressive sliding of undistorted planes. The measure of shear

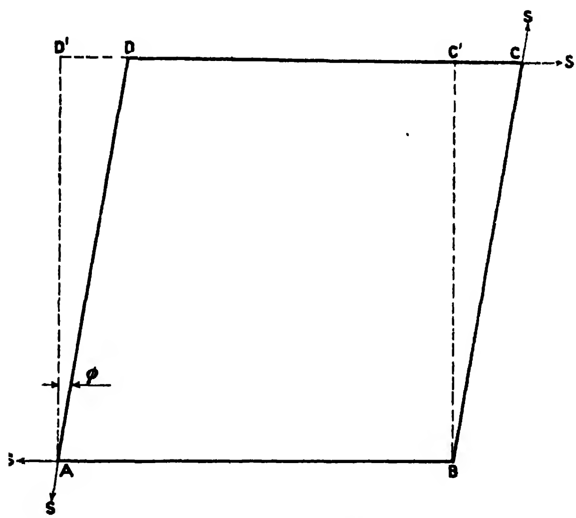


Fig. 6.—Cube in shear.

strain is then the amount of relative sliding of parallel planes a unit distance apart.

Thus, if opposite faces of a cube are subjected to equal shear stresses S, they will be distorted into rhombi, one of which is shown in Fig. 6. The angles ABC and ADC have increased and the angles BAD and BCD have decreased, the radian measure of the change ϕ, being known as the *shear strain*. Then, by *Hooke's law*, that "stress is proportional to strain,"

$$\phi = \frac{S}{E_s} \tag{45}$$

where ϕ = shear strain.
S = shear stress.
E_s = a constant called the *shear modulus* or the modulus of elasticity in shear of the material.

In practice, the angle ϕ is minute. It may therefore be replaced by ($\tan \phi$) or (DD'/AD'). Substituting (DD'/AD') for ϕ, Eq. (45) becomes

$$\frac{DD'}{AD'} = \frac{S}{E_s} \tag{46}$$

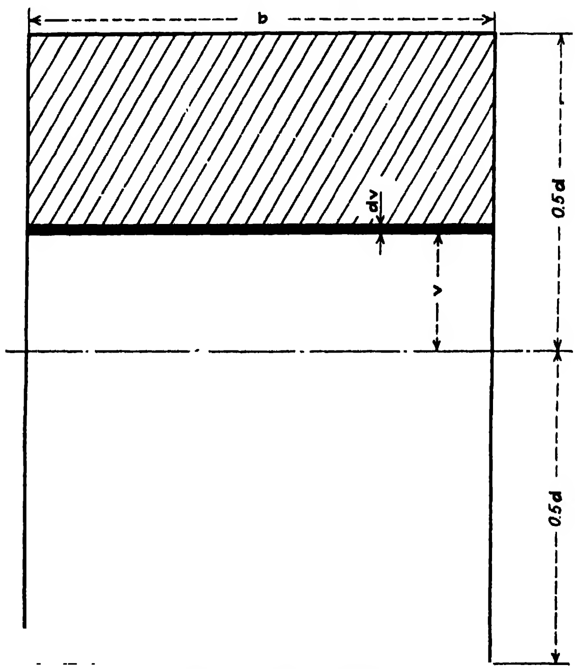


FIG. 7.—Beam in shear.

For all practical purposes the *shear modulus* E_s equals two-fifths of the value of *Young's modulus* E, or

$$E_s = \tfrac{2}{5}E \tag{47}$$

5.4. Deflection Due to Shear.

Figure 7 shows a section of a beam whose width is b, depth, d, and which is subjected to a shear V.

Let s_s = unit shearing stress at a distance v from the neutral axis.
I = moment of inertia of the section.
dA = area of an element.

Then, from the theory of beams,

$$s_s = \frac{V}{Ib}\int vdA \tag{48}$$

But, $dA = bdv$, and the upper and lower limits of integration are $d/2$ and v, respectively. Equation (48) may therefore be stated as

$$s_s = \frac{V}{I}\int_v^{\frac{d}{2}} vdv \tag{49}$$

Integrating Eq. (49),

$$s_s = \frac{V}{8I}(d^2 - 4v^2) \tag{50}$$

W'_i, the average internal work of shear per unit, equals one-half of the product of the unit stress s_s by the unit strain. But, by Eq. (46), unit strain equals s_s/E_s.

Therefore,

$$W'_i = \frac{s_s^{\ 2}}{2E_s} \tag{51}$$

Substituting the value of s_s [in Eq. (50)] into Eq. (51),

$$W'_i = \frac{V^2(d^4 - 8d^2v^2 + 16v^4)}{128I^2E_s} \tag{52}$$

Multiplying by an element of volume $bdvdx$, and integrating with respect to v, the upper and lower limits being $d/2$ and $-d/2$, respectively, the total internal work of shear, W_i, equals

$$W_i = \frac{V^2bd^5}{240I^2E_s}\int dx \tag{53}$$

For the section in Fig. 7, $I = bd^3/12$ and $V^2 = P^2$. Equation (53) therefore reduces to

$$W_i = \frac{3P^2L}{5bdE_s} \tag{54}$$

W_e, the external work done by P in deflecting the beam through the maximum shear deflection y_s equals one-half the product of P by y_s, or

$$W_e = \frac{Py_s}{2} \tag{55}$$

But $W_e = W_i$, and, therefore,

$$\frac{Py_s}{2} = \frac{3P^2L}{5bdE_s} \tag{56}$$

Also, $bd = A$ and $E_s = \frac{2}{5}E$. Substituting these values of bd and E_s into Eq. (56) and solving for y_s,

$$y_s = \frac{3PL}{AE} \tag{57}$$

where A is the area of the cross section.

5.5. Deflection Due to Moment and Shear.

The total maximum deflection Y of the beam in Fig. 5A is the sum of the moment and shear deflection. By Eqs. (44) and (57), therefore,

$$Y = y_m + y_s = \frac{PL^3}{12EI} + \frac{3PL}{AE} \tag{58}$$

But $A = I/r^2$, where r is the radius of gyration. Equation (58) may therefore be stated as

$$Y = \frac{PL(L^2 + 36r^2)}{12EI} \tag{59}$$

5.6. Effect of L/d on Moment and Shear Deflections.

Comparing Eqs. (44) and (57), the effect of the ratio L/d on the moment and shear deflections is

$$\frac{y_m}{y_s} = \frac{L^2}{3d^2}. \tag{60}$$

Equation (60) shows that where L/d is large, the shear deflection is negligible compared to the moment deflection, but, where the ratio is small, the shear deflection must be considered.

For example, if $L/d = 10$, then $y_m = 33.33y_s$, *i.e.*, the shear deflection is only 3 per cent of the moment deflection. If, on the other hand, $L/d = 1$, then $y_m = 0.33y_s$, *i.e.*, the shear deflection is 300 per cent of the moment deflection.

5.7. Deflections of Reinforced-concrete Beams.

The previous discussion applies only to beams of homogeneous, isotropic materials which obey Hooke's law, such as steel. Reinforced-concrete is neither homogeneous nor isotropic, and, although concrete in compression follows Hooke's law, concrete in tension does not do so, except for very low stresses. Equations (44), (57), (58), and (59) do not apply, therefore, to reinforced-concrete beams.

Turneaure and Maurer devised the following semiempirical moment-deflection formulas for reinforced-concrete beams. These give results which check reasonably well with measured deflections under temporary loads.

$$D_m = \frac{cWL^3n}{E_s bd^3K'} \tag{61}$$

$$K' = \frac{k^3 + (1 - k)^3 + 3np(1 - k)^2}{3} \tag{62}$$

$$k = \frac{1 + 2np}{2 + 2np} \tag{63}$$

where D_m = maximum deflection due to moment.

c = deflection constant in the equation for the deflection of a homogeneous beam, the equation being $c(WL^3/EI)$.

W = total load.

L = clear span.

n = ratio of Young's moduli for steel, E_s, and for concrete, E_c. The U. S. Navy Code recommends that $n = 8$ for use in the above formulas.

b = width of beam.

d = depth to center of tensile steel.

K' = numerical coefficient depending on p and n.

p = steel ratio.

k = proportionate depth of the neutral axis.

5.8. Moment Deflection of a Fixed-free but Guided Reinforced-concrete Beam.

For this case, $c = 1/12$, $W = P$. Equation (61) therefore becomes

$$D_m = \frac{2PL^3}{3E_s bd^3 K'} \tag{64}$$

But $bd^3 = 12\,I$. Equation (64) may therefore be expressed as

$$D_m = \frac{PL^3}{18E_s IK'} \tag{65}$$

5.9. Deflection Factor for Reinforced-concrete Beams.

Assuming a theoretical rectangular steel beam with end conditions and load as shown in Fig. 5A, the maximum moment deflection y'_m, by Eq. (44), is

$$y'_m = \frac{PL^3}{12E_s I} \tag{66}$$

Dividing Eq. (65) by Eq. (66),

$$D_m = \frac{y'_m}{1.5K'} \tag{67}$$

In other words, the maximum moment deflection of the steel beam, divided by

$$\text{Deflection factor} = (1.5K') \tag{68}$$

gives the maximum moment deflection of the reinforced-concrete beam.

5.10. Shear Deflection of a Reinforced-concrete Beam.

By analogy, the maximum shear deflection D_s of the reinforced-concrete beam equals the maximum shear deflection of the steel beam divided by the deflection factor, or

$$D_s = \frac{y'_s}{1.5K'} \tag{69}$$

5.11. Total Deflection of a Reinforced-concrete Beam.

The total maximum deflection D of the reinforced-concrete beam is obtained by adding the moment and shear deflections as given by Eqs. (67) and (69) or may be had

directly by dividing the total maximum deflection of the steel beam, as given by Eq. (59), by the deflection factor, or

$$D = \frac{PL(L^2 + 36r^2)}{18E_sIK'} \tag{70}$$

5.12. References.

BARTON, E. H.: "Analytical Mechanics."
BOYD, J. E.: "Strength of Materials."
NAITO, T.: "Earthquake Resisting Construction."
PRESCOTT, J.: "Applied Elasticity."
U. S. Bur. Standards, Tech. Paper 314.
U. S. Navy, Standards of Design for Concrete.

CHAPTER 6

FREE TRANSVERSE VIBRATIONS OF A SLENDER BEAM

6.1. Period of a Beam.

If a beam vibrates under the action of no external forces except such as are required to maintain the theoretical end conditions, the vibration is called *free.* Any free motion is some combination of the normal modes and is unaffected by gravity or by any other force which is independent of time. Of all the modes possible, the eye can ordinarily detect only the fundamental whose amplitude is the largest by far. The deflection curve in the fundamental mode differs very little from that which the beam assumes under static loading.

Equation (35) states that

$$\frac{d^4y}{dx^4} = -\frac{w}{EIg}\frac{d^2y}{dt^2} \tag{71}$$

But, by Eq. (7), $d^2y/dt^2 = -n^2y$. Substituting this value of the acceleration into Eq. (71),

$$\frac{d^4y}{dx^4} = \frac{n^2w}{EIg}y \tag{72}$$

Let

$$\frac{n^2w}{EIg} \tag{73}$$

then,

$$n^2 = \frac{m^4EIg}{w} \tag{74}$$

By Eq. (3), the free period T equals $2\pi/n$. Therefore,

$$n^2 = \frac{4\pi^2}{T^2} \tag{75}$$

Equating (74) to (75), and solving for T,

$$T = \frac{2\pi}{m^2}\sqrt{\frac{w}{EIg}} \tag{76}$$

6.2. Free Vibration Equations.

Substituting m^4 for its equivalent, Eq. (72) becomes

$$\frac{d^4y}{dx^4} = m^4y \tag{77}$$

the solution of which is

$$y = A\cosh mx + B\sinh mx + C\cos mx + D\sin mx \tag{78}$$

where y is the maximum deflection at any point x. The ratios $A:B:C:D$, and the real values of m and hence of n^2, are fixed by the conditions at each end of the beam.

The first derivative of y with respect to x gives the maximum slope at x, or

$$\frac{1}{m}\frac{dy}{dx} = A\sinh mx + B\cosh mx - C\sin mx + D\cos mx \tag{79}$$

The second derivative of y with respect to x gives the maximum bending moment at x, or

$$\frac{1}{m^2}\frac{d^2y}{dx^2} = A\cosh mx + B\sinh mx - C\cos mx - D\sin mx \tag{80}$$

The third derivative of y with respect to x gives the maximum shear at x, or

$$\frac{1}{m^3}\frac{d^3y}{dx^3} = A\sinh mx + B\cosh mx + C\sin mx - D\cos mx \tag{81}$$

6.3. Case 1. Free Vibrations of a Free-free Beam.

If a freely vibrating beam is free for rotation and translation at each end, the moments and shears at the ends are zero. Or, where $x = 0$ and $x = L$,

$$\frac{1}{m^2}\frac{d^2y}{dx^2} = 0 \text{ and } \frac{1}{m^3}\frac{d^3y}{dx^3} = 0.$$

Substituting 0 for x in Eqs. (80) and (81),

$$\frac{1}{m^2}\frac{d^2y}{dx^2} = 0 = A - C \tag{82}$$

$$\frac{1}{m^3}\frac{d^3y}{dx^3} = 0 = B - D \tag{83}$$

Substituting L for x in Eqs. (80) and (81),

$$\frac{1}{m^2}\frac{d^2y}{dx^2} = 0 = A \cosh mL + B \sinh mL - C \cos mL - D \sin mL \tag{84}$$

$$\frac{1}{m^3}\frac{d^3y}{dx^3} = 0 = A \sinh mL + B \cosh mL + C \sin mL - D \cos mL \tag{85}$$

By Eqs. (82) and (83), $A = C$ and $B = D$. Substituting A for C and B for D in Eqs. (84) and (85), solving each equation for A and equating the two forms of A, it is found that

$$\cos mL \cosh mL = 1 \tag{86}$$

The roots of Eq. (86) can be found graphically by plotting the curves $y = \cos mL$ and $y_1 = \frac{1}{\cosh mL}$, as shown in Fig. 8. The roots, as given by the abscissas of the points of intersection of the curves, are

$$\left.\begin{aligned} mL &= 4.73 \text{ (fundamental)} \\ m &= \frac{4.73}{L} \text{ (fundamental)} \end{aligned}\right\} \tag{87}$$

$$\left.\begin{aligned} m_1L &= 7.85 \text{ (first harmonic)} \\ m_1 &= \frac{7.85}{L} \text{ (first harmonic)} \end{aligned}\right\} \tag{88}$$

Substituting these values of m and m_1 into Eq. (76),

$$T = 0.28\sqrt{\frac{wL^4}{EIg}} \text{ (fundamental)} \tag{89}$$

and

$$T_1 = 0.10\sqrt{\frac{wL^4}{EIg}} \text{ (first harmonic)} \tag{90}$$

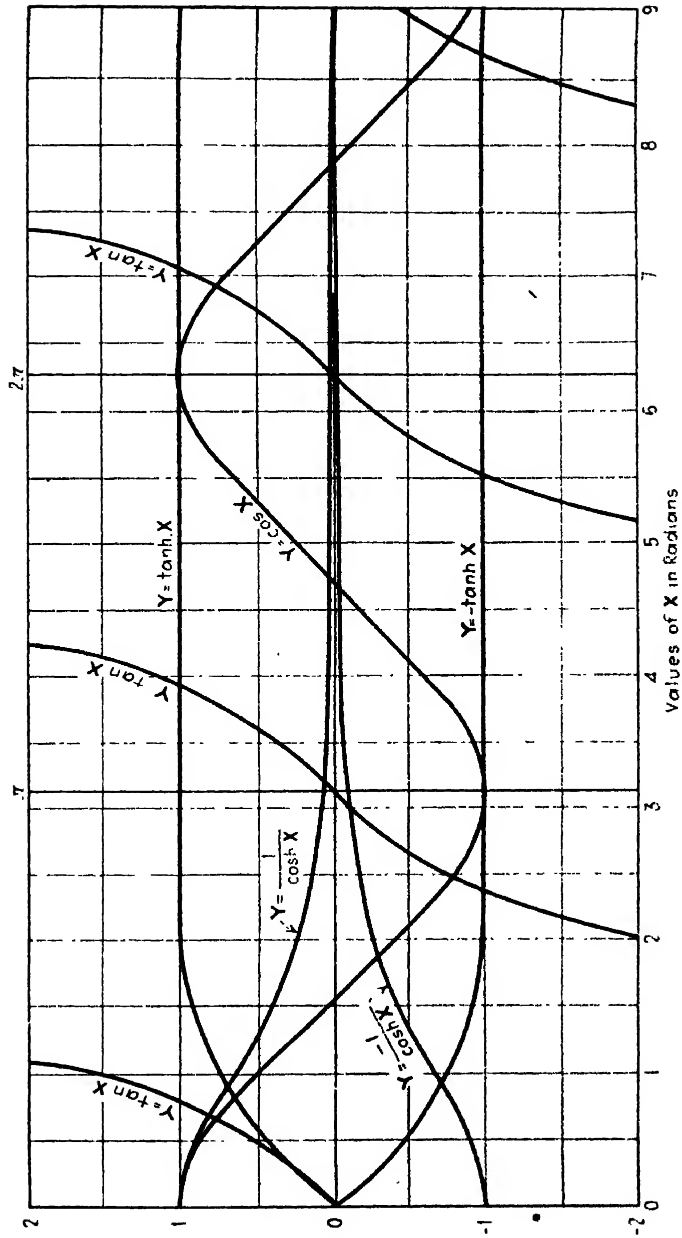

Y=tan X
Y=tan X
Y=tan X
Y=cos X
Y=tanh X
Y=-tanh X
Y=1/cosh X
Y=-1/cosh X
2π
π
Values of X in Radians

FIG. 8.

where w = weight per unit length.
L = length of beam.
E = Young's modulus of the material.
I = moment of inertia of section.
g = acceleration due to gravity = 32.2 ft. per second per second.

6.4. Case 2. Free Vibrations of a Hinged-free Beam.

As used here, the term *hinged* means free for rotation but fixed for translation. The conditions at the *hinged end* are:

$$x = 0, \qquad y = 0, \qquad \frac{1}{m^2}\frac{d^2y}{dx^2} = 0$$

Substituting 0 for x in Eqs. (78) and (80),

$$y = 0 = A + C \tag{91}$$

$$\frac{1}{m^2}\frac{d^2y}{dx^2} = 0 = C - A \tag{92}$$

The conditions at the *free end* are:

$$x = L, \qquad \frac{1}{m^2}\frac{d^2y}{dx^2} = 0, \qquad \frac{1}{m^3}\frac{d^3y}{dx^3} = 0$$

Substituting L for x in Eqs. (80) and (81),

$$\frac{1}{m^2}\frac{d^2y}{dx^2} = 0 = A \cosh mL + B \sinh mL - C \cos mL - D \sin mL \tag{93}$$

$$\frac{1}{m^3}\frac{d^3y}{dx^3} = 0 = A \sinh mL + B \cosh mL + C \sin mL - D \cos mL \tag{94}$$

By Eqs. (91) and (92), $A = 0 = C$. Substituting 0 for A and C in Eqs. (93) and (94), and dividing the first by the second, it is found that

$$\tan mL = \tanh mL \tag{95}$$

The roots of Eq. (95) can be found graphically by plotting the curves $y = \tan mL$ and $y_1 = \tanh mL$, as shown in Fig. 8.[4] The roots are:

$$\left.\begin{aligned} mL &= 3.93 \text{ (fundamental)} \\ m &= \frac{3.93}{L} \text{ (fundamental)} \end{aligned}\right\} \tag{96}$$

$$\left.\begin{aligned} m_1L &= 7.07 \text{ (first harmonic)} \\ m_1 &= \frac{7.07}{L} \text{ (first harmonic)} \end{aligned}\right\} \tag{97}$$

Substituting these values of m and m_1 into Eq. (76),

$$T = 0.41\sqrt{\frac{wL^4}{EIg}} \text{ (fundamental)} \tag{98}$$

$$T_1 = 0.13\sqrt{\frac{wL^4}{EIg}} \text{ (first harmonic)} \tag{99}$$

6.5. Case 3. Free Vibrations of a Sliding-free Beam.

As used here, the term *sliding* means free for translation but fixed for rotation. The conditions at the *sliding end* are:

$$x = 0, \qquad \frac{1}{m}\frac{dy}{dx} = 0, \qquad \frac{1}{m^3}\frac{d^3y}{dx^3} = 0$$

Substituting 0 for x in Eqs. (79) and (81),

$$\frac{1}{m}\frac{dy}{dx} = 0 = B + D \tag{100}$$

$$\frac{1}{m^3}\frac{d^3y}{dx^3} = 0 = B - D \tag{101}$$

The conditions at the *free end* are:

$$x = L, \qquad \frac{1}{m^2}\frac{d^2y}{dx^2} = 0, \qquad \frac{1}{m^3}\frac{d^3y}{dx^3} = 0$$

Substituting L for x in Eqs. (80) and (81),

$$\frac{1}{m^2}\frac{d^2y}{dx^2} = 0 = A \cosh mL + B \sinh mL - C \cos mL - D \sin mL \tag{102}$$

$$\frac{1}{m^3}\frac{d^3y}{dx^3} = 0 = A \sinh mL + B \cosh mL + C \sin mL - D \cos mL) \tag{103}$$

By Eqs. (100) and (101), $B = 0 = D$. Substituting 0 for B and D in Eqs. (102) and (103), solving each for A, equating the two forms of A, and reducing, it is found that

$$\tan mL = -\tanh mL \tag{104}$$

The roots of Eq. (104) can be found graphically by plotting the curves $y = \tan mL$ and $y_1 = -\tanh mL$, as shown in Fig. 8. The roots are:

$$\left.\begin{aligned} mL &= 2.37 \text{ (fundamental)} \\ m &= \frac{2.37}{L} \text{ (fundamental)} \end{aligned}\right\} \tag{105}$$

$$\left.\begin{aligned} m_1L &= 5.50 \text{ (first harmonic)} \\ m_1 &= \frac{5.50}{L} \text{ (first harmonic)} \end{aligned}\right\} \tag{106}$$

Substituting these values of m and m_1 into Eq. (76),

$$T = 1.12\sqrt{\frac{wL^4}{EIg}} \text{ (fundamental)} \tag{107}$$

$$T_1 = 0.21\sqrt{\frac{wL^4}{EIg}} \text{ (first harmonic)} \tag{108}$$

6.6. Case 4. Free Vibrations of a Fixed-free Beam.

As used here, the term *fixed* means fixed for translation and rotation. The conditions at the *fixed end* are:

$$x = 0, \qquad y = 0, \qquad \frac{1}{m}\frac{dy}{dx} = 0$$

Substituting 0 for x in Eqs. (78) and (79),

$$y = 0 = A + C \tag{109}$$

$$\frac{1}{m}\frac{dy}{dx} = 0 = B + D \tag{110}$$

The conditions at the *free end* are:

$$x = L, \qquad \frac{1}{m^2}\frac{d^2y}{dx^2} = 0, \qquad \frac{1}{m^3}\frac{d^3y}{dx^3} = 0$$

Substituting L for x in Eqs. (80) and (81),

$$\frac{1}{m^2}\frac{d^2y}{dx^2} = 0 = A \cosh mL + B \sinh mL - C \cos mL - D \sin mL \qquad (111)$$

$$\frac{1}{m^3}\frac{d^3y}{dx^3} = 0 = A \sinh mL + B \cosh mL + C \sin mL - D \cos mL \qquad (112)$$

By Eqs. (109) and (110), $A = -C$ and $B = -D$. Substituting A for $-C$ and B for $-D$ in Eqs. (111) and (112), solving each for A, equating the two forms of A, and reducing it is found that

$$\cos mL \cosh mL = -1 \qquad (113)$$

The roots of Eq. (113) are found graphically by plotting the curves $y = \cos mL$ and $y_1 = \frac{-1}{\cosh mL}$, as shown in Fig. 8. The roots are:

$$\left.\begin{aligned} mL &= 1.88 \text{ (fundamental)} \\ m &= \frac{1.88}{L} \text{ (fundamental)} \end{aligned}\right\} \qquad (114)$$

$$\left.\begin{aligned} m_1L &= 4.69 \text{ (first harmonic)} \\ m_1 &= \frac{4.69}{L} \text{ (first harmonic)} \end{aligned}\right\} \qquad (115)$$

Substituting these values of m and m_1 into Eq. (76),

$$T = 1.79\sqrt{\frac{wL^4}{EIg}} \text{ (fundamental)} \qquad (116)$$

$$T_1 = 0.29\sqrt{\frac{wL^4}{EIg}} \text{ (first harmonic)} \qquad (117)$$

6.7. Transverse vs. Longitudinal Periods of Beams.

When a homogeneous, isotropic beam is acted on by impact in a longitudinal direction, it is thrown into free longitudinal vibration. Poisson showed that the relation between the transverse and longitudinal periods is

$$T = \frac{T'L}{\pi(mL)^2r} \qquad (118)$$

where T = transverse period of vibration.
T' = longitudinal period of vibration.
L = length of beam.
mL = constant depending on the end conditions.
r = radius of gyration.

6.8. Effect of the End Conditions on the Periods.

Tabulating the numerical coefficients of the fundamental and first harmonic periods in cases 1 to 4 inclusive, and obtaining the ratios of the fundamentals to the first harmonics, the following schedule results:

End conditions	(1) Coefficient of T (fundamental)	(2) Coefficient of T_1 (first harmonic)	(3) T/T_1
Fixed-free	1.79	0.29	6.2
Sliding-free	1 12	0.21	5.3
Hinged-free	0 41	0.13	3.2
Free-free	0.28	0.10	2.8

Column 1 shows that the fundamental period of a fixed-free beam is 1.6 times as large as that of a similar but sliding-free beam, 4.4 times as large as that of a similar but hinged-free beam, and 6.4 times as large as that of a similar but free-free beam.

Column 2 shows that the first harmonic period of a fixed-free beam is 1.4 times as large as that of a similar but sliding-free beam, 2.2 times as large as that of a similar but hinged-free beam, and 2.9 times as large as that of a similar but free-free beam.

Consequently, if the fundamental or harmonics of a beam lie within the bracket of periods of destructive earthquakes mentioned in Par. 3.4, a simple but effective method of changing the periods of the beam consists in changing its end conditions. For example, if the fundamental period of a fixed-free beam is 1.0 sec., the dangerous bracket may be avoided by making the end conditions either sliding-free, hinged-free or free-free, with periods of 0.62, 0.23, and 0.16

sec., respectively. Again, if the fundamental period of a free-free beam is 1.2 sec., synchronism is avoided by changing to either hinged-free, sliding-free, or fixed-free end conditions, the fundamental periods becoming 1.76, 4.80, and 7.68 sec., respectively.

In the last case, however, the first harmonic of the fixed-free beam has a value of 1.25 sec. Only the hinged-free and sliding-free beams, therefore, satisfy requirements.

If the end conditions of a beam are unknown but the periods are known, it is possible to determine the end conditions by means of the ratio T/T_1 and Column 3. Thus, if $T = 0.81$ sec. and $T_1 = 0.13$ sec., then $T/T_1 = 6.2$. Referring to Column 3, it is seen that the end conditions are fixed-free.

6.9. Concrete vs. Steel in Damping Free Vibrations.

To determine the relative values of concrete and steel in damping free vibrations, rectangular, fixed-free, steel, and reinforced-concrete beams will be compared. To simplify calculations, the beams assumed were designed to carry equal loads and are identical in length and width but have different depths.

The fundamental period of a fixed-free beam is given by Eq. (116), which when squared becomes

$$T^2 = 3.20\frac{wL^4}{EIg} \tag{119}$$

Therefore, if the subscripts c and s denote concrete and steel, respectively,

$$\frac{T_c^2}{T_s^2} = \frac{w_cE_sI_s}{w_sE_cI_c} \tag{120}$$

Let M = bending moment at any point x.

w = weight per linear foot.

s = extreme fiber stress in compression.

c = one-half the depth d.

b = width of the beam.

Then, for a uniformly loaded, rectangular, fixed-free beam,

$$M = \frac{wx^2}{2}, \qquad s = \frac{Mc}{I}, \qquad I = \frac{bd^3}{12} = \frac{2bc^3}{3}$$

and

$$c^3 = \left(\frac{3wx^2}{4bs}\right)^{3/2} \tag{121}$$

Substituting their equivalents for I_c, I_s, c_c^3, and c_s^3, Eq. (120) becomes,

$$\frac{T_c^2}{T_s^2} = \frac{E_s}{E_c}\left(\frac{w_s}{w_c}\right)^{1/2}\left(\frac{s_c}{s_s}\right)^{3/2} \tag{122}$$

Suyehiro showed that the damping value of a fixed-free beam is covered by Eq. (20), in which the damping factor is

$$e^{-\frac{1}{2}kt} \tag{123}$$

But, by Par. 4.6 and Eq. (3),

$$k = \frac{R}{M}, \qquad M = \frac{K}{n^2}, \qquad n^2 = \frac{(2\pi)^2}{T^2} \tag{124}$$

Also, within the elastic limit, K may be replaced by Young's modulus E. Therefore,

$$k = \frac{(2\pi)^2}{T^2}\frac{R}{E} \tag{125}$$

and

$$\frac{T_c^2}{T_s^2} = \frac{E_s k_s R_c}{E_c k_c R_s} \tag{126}$$

Equating (122) to (126),

$$\frac{k_c}{k_s} = \frac{R_c}{R_s}\left(\frac{w_c}{w_s}\right)^{1/2}\left(\frac{s_s}{s_c}\right)^{3/2} \tag{127}$$

The following values of R are given by Suyehiro:

$$R_c = 7270, \qquad R_s = 164{,}650$$

The allowable extreme fiber stresses in compression under the U. S. Navy Code are:

For 2000 lb. concrete,

$$s_c = 700 \text{ lb. per square inch}$$

and for structural steel,

$$s_s = 18{,}000 \text{ lb. per square inch}$$

Since the beams were designed to carry identical loads, the maximum resisting moments are equal. Therefore,

$$\frac{s_c I_c}{c_c} = \frac{s_s I_s}{c_s}$$

and

$$c_c = 5.07 c_s$$

Assuming that $b = 1$ ft. 00 in. and that $d_s = 1$ ft. 00 in., then the depth of the concrete beam will be $d_c = 5.07$ ft. The weight per linear foot of each beam is,

$$w_s = 490 \text{ lb. per foot}$$

and

$$w_c = 5.07 \times 150 = 760 \text{ lb. per foot}$$

Substituting these values into Eq. (127), and solving for k_c,

$$k_c = 7.1 k_s \tag{128}$$

By (123), therefore, the damping factor of the steel beam is

$$e^{-\frac{1}{2}k_s t}$$

whereas for the reinforced-concrete beam it is,

$$e^{-\frac{1}{2}(7.1\,k_s)t}$$

Comparing the two damping factors, it is obvious that reinforced concrete is far more effective than steel in damping free vibrations. The reinforced-concrete beam will come to rest much sooner than the steel beam, and its abnormal amplitude during synchronism will be less.

6.10. Summary of Free Vibrations of Beams.

Equation (76) may be expressed as,

$$T = \frac{2\pi}{c^2} L^2 \sqrt{\frac{w}{EIg}}$$

where T = free period of vibration.
c = constant depending on the end conditions.
L = length of beam.
w = weight per unit length.
E = Young's modulus of the material.
I = moment of inertia = Ar^2; A = sectional area, and r = radius of gyration.
g = acceleration due to gravity = 32.2 ft. per second per second.

It is obvious that the period varies directly as L^2, directly as the square root of w, inversely as the square roots of E, I, and g, inversely as r, and inversely as the three-half power of the depth in the plane of vibration, provided that the depth d is small compared to the length L. If d is large compared to L, the true period T_t can be found by means of Eq. (129) below, which was obtained as follows:

Equation (76) may be stated as

$$T^2 = k\frac{wL^4}{EI}$$

where T is the free period of a beam whose L/d is large and whose static deflection is therefore almost entirely due to moment.

Also, the maximum static moment deflection of a beam may be given as

$$y_m = c_1\frac{wL^4}{EI}$$

where c_1 is a static deflection constant.

Therefore,

$$y_m = \frac{T^2}{k_2}$$

and

$$T = (k_2 y_m)^{1/2}$$

That is to say, that the free period of a beam whose L/d is large is proportional to the square root of its maximum moment deflection.

Assuming, therefore, that the period of a beam whose L/d is small is proportional to the square root of its total maximum deflection Y, then, since $Y = y_m + y_s$,

$$T_t = \sqrt{k_2(y_m + y_s)}$$

where y_s is the maximum static deflection due to shear, and T_t is the true period.

By Eq. (57), substituting wL for P, for uniform load,

$$y_s = \frac{3wL^2}{AE}$$

Therefore, since $A = bd$ and $I = bd^3/12$,

$$y_s = y_m\left(\frac{d^2}{4c_1L^2}\right)$$

Substituting its equivalent for y_s, and (T^2/k_2) for y_m,

$$T_t = \sqrt{T^2\left(1 + \frac{d^2}{4c_1L^2}\right)} \tag{129}$$

6.11. References.

LAMB, H.: "Dynamical Theory of Sound."

LOVE, A. E. H.: "A Treatise on the Mathematical Theory of Elasticity."

MORROW, J.: Lateral Vibration of Bars of Uniform and Varying Cross-Section, *Phil. Mag.*, 1905.

PRESCOTT, J.: "Applied Elasticity."

RAYLEIGH, LORD: "Theory of Sound," vol. 1.

SUYEHIRO, K.: Engineering Seismology, *Proc. Amer. Soc. Civil Eng.*, May, 1932; *Bull. Imp. Earthquake Research Inst. Japan*, vol. 6, p. 63.

TIMOSHENKO, S.: "Applied Elasticity."

———: "Vibration Problems in Engineering."

CHAPTER 7

FORCED TRANSVERSE VIBRATIONS OF A SLENDER BEAM

7.1. Forced vs. Free Vibrations.

By Eq. (35),

$$\frac{d^4y}{dx^4} = -\frac{w}{EIg}\frac{d^2y}{dt^2} \qquad (130)$$

During a forced simple harmonic vibration:

$y = \phi(x)f(t) =$ deflection of any point x at any time t.

$\phi(x) =$ function of $x = A \cosh m'x + B \sinh m'x + C \cos m'x + D \sin m'x$.

$f(t) =$ function of $t = \cos pt + r \sin pt$. The first term, $\cos pt$, represents the initial displacement, and the second term, $r \sin pt$, takes care of the initial velocity. It was shown in Par. 3.7 that the total seismic velocity of a particle is low. The initial velocity will therefore be negligible. Assuming that the initial velocity is zero, $f(t) = \cos pt$, with a maximum value of $f(t) = 1$, when the phase angle $pt = 0$.

The maximum value of y therefore occurs when $y = \phi(x)$.

Impressing a forced vibration,

$$y = \phi(x)f(t) = F \cos pt \quad (\text{at } x = 0) \qquad (131)$$

where $\phi(x) = F$.

$f(t) = \cos pt$.

$F =$ amplitude.

$p =$ forced angular velocity.

$t =$ time.

Then, the forced period T_p, by Eq. (3), equals $2\pi/p$, and

$$T_p^2 = \frac{(2\pi)^2}{p^2} \qquad (132)$$

By Eq. (7), the acceleration at any point, d^2y/dt^2, equals $-p^2y$.

Substituting $-p^2y$ for d^2y/dt^2 in Eq. (130),

$$\frac{d^4y}{dx^4} = \frac{p^2w}{EIg}y \tag{133}$$

Let

$$m'^4 = \frac{p^2w}{EIg} \tag{134}$$

then,

$$\frac{m'^4EIg}{w} \tag{135}$$

Equating (132) to (135), and solving for T_p,

$$T_p = \frac{2\pi}{m'^2}\sqrt{\frac{w}{EIg}} \tag{136}$$

But, by Eq. (76),

$$T = \frac{2\pi}{m^2}\sqrt{\frac{w}{EIg}} \tag{137}$$

Dividing Eq. (137) by (136), solving for m', and multiplying both sides of the equation by L, it is found that

$$m'L = mL\sqrt{\frac{T}{T_p}} \tag{138}$$

But mL is a constant of free vibration which depends on the end conditions of a beam. $m'L$, the constant of

End condition at $x = 0$	Vibration	Deflection	Slope	Moment	Shear
Fixed	Free	0	0	?	
	Forced	F	0	?	
Sliding	Free		0	?	
	Forced		0	?	
Hinged	Free	0	?	0	
	Forced	F	?	0	
Free	Free	?	?	0	
	Forced	F	?	0	

F = amplitude; ? = unknown; 0 = zero.

forced vibration, can therefore be computed in terms of the ratio between the free and forced periods of vibration.

In the table shown on page 55 the end conditions, at $x = 0$, are shown for the free and forced vibrations of beams. The end conditions, at $x = L$, are identical for all cases, the deflections and slopes being unknown and the moments and shears being zero.

7.2. Forced-vibration Equations.

Substituting m'^4 for p^2w/EIg, in Eq. (133),

$$\frac{d^4y}{dx^4} = m'^4y \tag{139}$$

the solution of which is

$$y = A \cosh m'x + B \sinh m'x + C \cos m'x + D \sin m'x \tag{140}$$

where y is the maximum deflection at any point x. The ratios $A:B:C:D$, and the real values of m' and hence of p^2, are determined by the conditions at each end of the beam.

The first derivative of y with respect to x gives the maximum slope at any point, or

$$\frac{1}{m'}\frac{dy}{dx} = A \sinh m'x + B \cosh m'x - C \sin m'x + D \cos m'x \tag{141}$$

The second derivative of y with respect to x gives the maximum bending moment at any point, or

$$\frac{1}{m'^2}\frac{d^2y}{dx^2} = A \cosh m'x + B \sinh m'x - C \cos m'x - D \sin m'x \tag{142}$$

The third derivative of y with respect to x gives the maximum shear at any point, or

$$\frac{1}{m'^3}\frac{d^3y}{dx^3} = A \sinh m'x + B \cosh m'x + C \sin m'x - D \cos m'x \tag{143}$$

7.3. Case 1. Forced Vibrations of a Fixed-free Beam.

For a fixed-free beam the conditions at the *fixed end* are

$$x = 0, \qquad y = F, \qquad \frac{1}{m'}\frac{dy}{dx} = 0$$

Substituting 0 for x in Eqs. (140) and (141),

$$y = F = A + C \tag{144}$$

$$\frac{1}{m'}\frac{dy}{dx} = 0 = B + D \tag{145}$$

The conditions at the *free end* are

$$x = L, \qquad \frac{1}{m'^2}\frac{d^2y}{dx^2} = 0, \qquad \frac{1}{m'^3}\frac{d^3y}{dx^3} = 0$$

Substituting L for x in Eqs. (142) and (143),

$$\frac{1}{m'^2}\frac{d^2y}{dx^2} = 0 = A \cosh m'L + B \sinh m'L - C \cos m'L - D \sin m'L \tag{146}$$

$$\frac{1}{m'^3}\frac{d^3y}{dx^3} = 0 = A \sinh m'L + B \cosh m'L + C \sin m'L - D \cos m'L \tag{147}$$

By Eqs. (144) and (145), $A = F - C$ and $B = -D$. Substituting $F - C$ for A and $-D$ for B in Eqs. (146) and (147),

$$F \cosh m'L - C \cosh m'L - D \sinh m'L - C \cosh m'L - D \sin m'L = 0 \tag{148}$$

$$F \sinh m'L - C \sinh m'L - D \cosh m'L + C \sin m'L - D \cos m'L = 0 \tag{149}$$

Solving Eqs. (148) and (149) for C, equating the two forms of C, and solving for D:

$$D = \frac{F}{2}\frac{(\sinh m'L \cos m'L + \cosh m'L \sin m'L)}{(1 + \cosh m'L \cos m'L)} \tag{150}$$

$$B = -D \tag{151}$$

$$C = \frac{F}{2}\frac{(1 - \sinh m'L \sin m'L + \cosh m'L \cos m'L)}{(1 + \cosh m'L \cos m'L)} \tag{152}$$

$$A = \frac{F}{2}\frac{(1 + \sinh m'L \sin m'L + \cosh m'L \cos m'L)}{(1 + \cosh m'L \cos m'L)} \tag{153}$$

The maximum deflection, moment, or shear at any point x is found by substituting the values of A, B, C, D, and x into Eqs. (140), (142), and (143).

Thus, by Eq. (140), the maximum deflection, or amplitude, where $x = 0$, is

$$y = A + C = F \tag{154}$$

By Eq. (142), the maximum moment, where $x = 0$, is

$$\frac{1}{m'^2}\frac{d^2y}{dx^2} = A - C \tag{155}$$

But Eq. (31) defines bending moment M as $EI\frac{d^2y}{dx^2}$. Therefore,

$$EI\frac{d^2y}{dx^2} = EIm'^2(A - C) \tag{156}$$

By Eq. (143), the maximum shear, where $x = 0$, is

$$\frac{1}{m'^3}\frac{d^3y}{dx^3} = B - D \tag{157}$$

But Eq. (32) defines shear V as $EI\frac{d^3y}{dx^3}$. Therefore,

$$EI\frac{d^3y}{dx^3} = EIm'^3(B - D) \tag{158}$$

7.4. Influence Curves for Case 1.

By Eqs. (114) and (138),

$$m'L = 1.88\sqrt{\frac{T}{T_p}} \tag{159}$$

Let

$$x = cL$$

then,

$$m'x = cm'L$$

Therefore,

$$m'x = cm'L = c\left(1.88\sqrt{\frac{T}{T_p}}\right) \tag{160}$$

Assuming that $F = 1$, the maximum unit deflections Y_u, unit shears V_u, and unit moments M_u were calculated for

a number of values of c and T/T_p by means of Eqs. (140), (142), and (143). The results are shown below, the influence curves in Figs. 9, 10, and 11 being plotted therefrom.

The actual maximum deflection y is

$$y = FY_u \tag{161}$$

Y_u being obtained from the correct T/T_p curve in Fig. 9, and F being the amplitude at $x = 0$.

The actual maximum moment M at any point x is

$$M = EI\frac{d^2y}{dx^2} = EIm'^2FM_u \tag{162}$$

M_u being obtained from the correct T/T_p curve in Fig. 10.

The actual maximum shear V at any point x is

$$V = EI\frac{d^3y}{dx^3} = EIm'^3FV_u \tag{163}$$

V_u being obtained from the correct T/T_p curve in Fig. 11.

FORCED VIBRATIONS OF A FIXED-FREE BEAM
($F = 1$)

T/T_p	$m'x$	Y_u	M_u	V_u	c	T/T_p	$m'x$	Y_u	M_u	V_u
0.5	0.00	1.00	1.16	−1.60	0.0	0.7	0.00	1.00	2.30	−2.52
	0.27	1.03	0.77	−1.33	0.2		0.31	1.10	1.56	−2.19
$A =$ 1.08	0.53	1.13	0.45	−1.05	0.4	$A =$ 1.65	0.63	1.36	0.94	−1.82
$B =$ −0.80	0.80	1.25	0.23	−0.73	0.6	$B =$ −1.26	0.94	1.71	0.43	−1.34
$C =$ −0.08	1.06	1.39	0.07	−0.38	0.8	$C =$ −0.85	1.26	2.09	0.09	−0.85
$D =$ 0.80	1.33	1.53	0.00	0.00	1.0	$D =$ 1.26	1.57	2.50	0.00	0.00
0.9	0.00	1.00	7.84	−6.60	0.0	1.1	0.00	1.00	−7.54	4.66
	0.36	1.45	5.55	−6.20	0.2		0.39	0.45	−5.63	5.00
$A =$ 4.42	0.71	2.60	3.46	−5.49	0.4	$A =$ −3.27	0.79	−0.94	−3.66	4.88
$B =$ −3.30	1.07	4.23	1.73	−4.31	0.6	$B =$ 2.33	1.18	−2.91	−1.83	4.13
$C =$ −3.42	1.42	6.02	0.50	−2.45	0.8	$C =$ 4.27	1.58	−5.22	−0.52	2.59
$D =$ 3.30	1.78	7.93	0.00	0.00	1.0	$D =$ −2.33	1.97	−7.59	0.00	0.00
1.5	0.00	1.00	−1.56	0.20	0.0	3.0	0.00	1.00	0.12	−1.21
	0.46	0.85	−1.35	0.64	0.2		0.65	0.97	−0.47	−0.57
$A =$ −0.28	0.92	0.40	−1.00	0.93	0.4	$A =$ 0.56	1.30	0.78	−0.64	0.01
$B =$ 0.10	1.38	−0.26	−0.54	0.97	0.6	$B =$ −0.61	1.95	0.31	−0.49	0.38
$C =$ 1.28	1.84	−1.04	−0.16	0.66	0.8	$C =$ 0.44	2.61	−0.36	−0.22	0.37
$D =$ −0.10	2.30	−1.84	0.00	0.00	1.0	$D =$ 0.61	3.26	−1.00	0.00	0.00

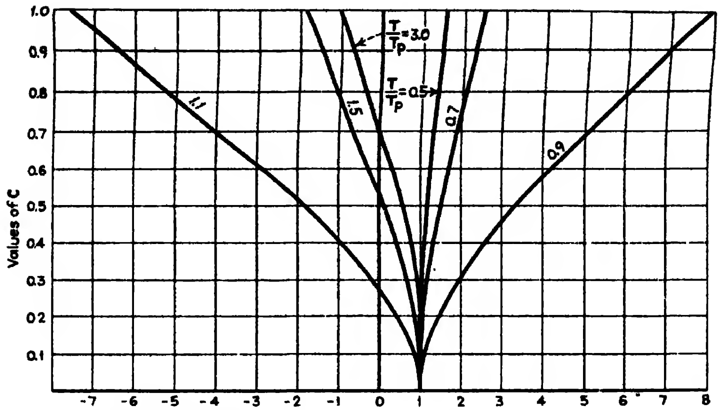


FIG. 9.—Values of Y_u—fixed free beam, forced vibrations.

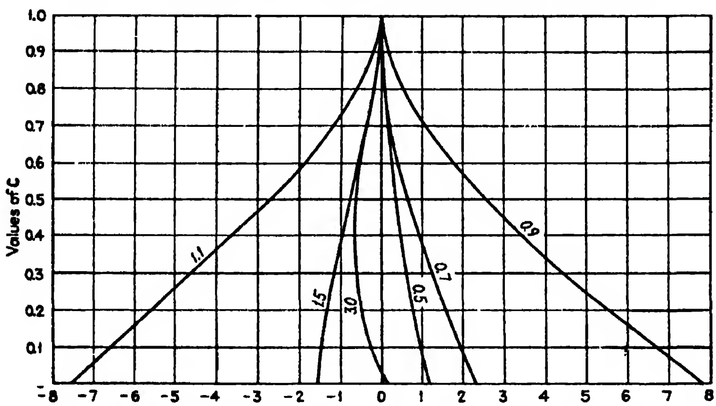


FIG. 10.—Values of M_u—fixed-free beam, forced vibrations.

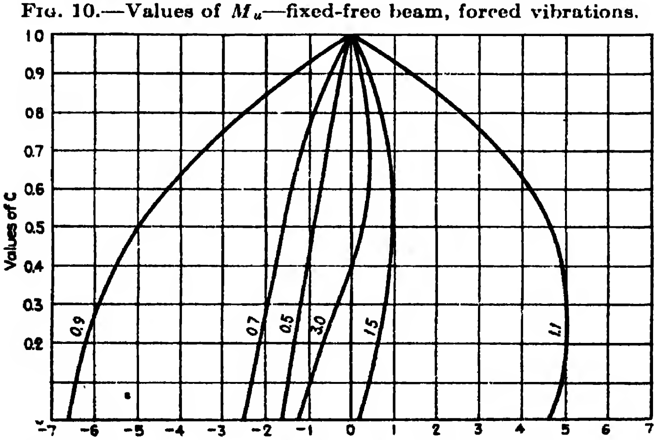


FIG. 11.—Values of V_u—fixed-free beam, forced vibrations.

7.5. Case 2. Forced Vibrations of a Sliding-free Beam.

For a sliding-free beam the conditions at the *sliding end* are

$$x = 0, \qquad y = F, \qquad \frac{1}{m'}\frac{dy}{dx} = 0$$

It should be noted that the shear, at $x = 0$, can no longer be assumed as zero, since the assigned motion is being accomplished by a periodic force at that point.

The conditions at the *free end* are

$$x = L, \qquad \frac{1}{m'^2}\frac{d^2y}{dx^2} = 0, \qquad \frac{1}{m'^3}\frac{d^3y}{dx^3} = 0$$

The end conditions for case 2 are therefore qualitatively the same as for case 1. The solutions are identical. Figures 9, 10, 11, and Eqs. (161), (162), and (163) apply to both cases.

It should be kept in mind, however, that the free periods for the two cases differ. Consequently, for beams identical, except for the end conditions, the deflections, moments, and shears will differ quantitatively.

7.6. Case 3. Forced Vibrations of a Hinged-free Beam.

For a hinged-free beam, the conditions at the *hinged end* are

$$x = 0, \qquad y = F, \qquad \frac{1}{m'^2}\frac{d^2y}{dx^2} = 0$$

Substituting 0 for x in Eqs. (140) and (142),

$$y = F = A + C \tag{164}$$

$$\frac{1}{m'^2}\frac{d^2y}{dx^2} = 0 = C - A \tag{165}$$

The conditions at the *free end* are

$$x = L, \qquad \frac{1}{m'^2}\frac{d^2y}{dx^2} = 0, \qquad \frac{1}{m'^3}\frac{d^3y}{dx} = 0$$

Substituting L for x in Eqs. (142) and (143),

$$\frac{1}{m'^2}\frac{d^2y}{dx^2} = 0 = A\cosh m'L + B\sinh m'L - C\cos m'L$$
$$D\sin m'L \tag{166}$$

$$\frac{1}{m'^3}\frac{d^3y}{dx^3} = 0 = A \sinh m'L + B \cosh m'L + C \sin m'L - D \cos m'L \qquad (167)$$

By Eqs. (164) and (165), $A = C = F/2$. Substituting $F/2$ for A and C, in Eqs. (166) and (167),

$$\frac{F}{2} \cosh m'L + B \sinh m'L - \frac{F}{2} \cos m'L - D \sin m'L = 0 \qquad (168)$$

$$\frac{F}{2} \sinh m'L + B \cosh m'L + \frac{F}{2} \sin m'L - D \cos m'L = 0 \qquad (169)$$

Solving Eqs. (168) and (169) for D, equating the two forms of D, and solving for B,

$$B = \frac{F}{2}\frac{(\sinh m'L \sin m'L - \cosh m'L \cos m'L + 1)}{(\sinh m'L \cos m'L - \cosh m'L \sin m'L)} \qquad (170)$$

$$D = \frac{F}{2}\frac{(\sinh m'L \sin m'L + \cosh m'L \cos m'L - 1)}{(\sinh m'L \cos m'L - \cosh m'L \sin m'L)} \qquad (171)$$

$$A = C = \frac{F}{2} \qquad (172)$$

The maximum deflection, moment, or shear at any point x is found by substituting the values of A, B, C, D, and x into Eqs. (140), (142), and (143).

Thus, by Eq. (140), the maximum deflection, at $x = 0$, is

$$y = A + C = F \qquad (173)$$

By Eq. (142), the maximum moment, at $x = 0$, is

$$\frac{1}{m'^2}\frac{d^2y}{dx^2} = A - C \qquad (174)$$

But Eq. (31) defines bending moment M as $EI\frac{d^2y}{dx^2}$. Therefore,

$$EI\frac{d^2y}{dx^2} = EIm'^2(A - C) \qquad (175)$$

By Eq. (143), the maximum shear, at $x = 0$, is

$$\frac{1}{m'^3}\frac{d^3y}{dx^3} = B - D \qquad (176)$$

But Eq. (32) defines shear V as $EI\dfrac{d^3y}{dx^3}$. Therefore,

$$EI\frac{d^3y}{dx^3} = EIm'^3(B - D) \tag{177}$$

7.7. Influence Curves for Case 3.

By Eqs. (96) and (138),

$$m'L = 3.93\sqrt{\frac{T}{T_p}} \tag{178}$$

Also, as in Par. 7.4,

$$m'x = cm'L = c\left(3.93\sqrt{\frac{T}{T_p}}\right) \tag{179}$$

Assuming that $F = 1$, the maximum unit deflections Y_u, moments M_u, and shear V_u were calculated for a number of values of c and T/T_p by means of Eqs. (140), (142), and (143). The results are shown below, the influence curves in Fig. 12 being plotted therefrom.

The actual maximum deflections y, moments M, and shears V at any point x are

$$y = FY_u \tag{180}$$

$$M = EIm'^2FM_u \tag{181}$$

$$V = EIm'^3FV_u \tag{182}$$

F being the amplitude at $x = 0$, and Y_u, M_u, and V_u being obtained from the proper T/T_p curves in Fig. 12.

Forced Vibrations of a Hinged-free Beam

($F = 1$)

T/T_p	$m'x$	Y_u	M_u	V_u	c	T/T_p	$m'x$	Y_u	M_u	V_u
0.5	0.00	1.00	0.00	−0.83	0.0	0.9	0.00	1.00	0.00	−3.21
	0.56	0.82	−0.32	−0.33	0.2		0.75	2.32	−1.98	−1.93
$A =$ 0.50	1.11	0.56	−0.38	0.07	0.4	$A =$ 0.50	1.49	2.59	−2.73	−0.03
$B =$ −0.55	1.67	0.20	−0.26	0.30	0.6	$B =$ −0.59	2.24	1.39	−2.11	1.53
$C =$ 0.50	2.22	−0.26	−0.10	0.28	0.8	$C =$ 0.50	2.98	−0.93	0.77	1.74
$D =$ 0.28	2.78	−0.73	0.00	0.00	1.0	$D =$ 2.62	3.73	−3.71	0.00	0.00
1.1	0.00	1.00	0.00	2.16	0.0	1.5	0.00	1.00	0.00	−0.07
	0.82	−1.29	1.85	1.99	0.2		0.96	0.15	0.25	0.47
$A =$ 0.50	1.65	−2.38	2.88	0.36	0.4	$A =$ 0.50	1.92	−0.46	0.66	0.28
$B =$ −0.44	2.47	−1.59	2.43	−1.43	0.6	$B =$ −0.49	2.89	−0.50	0.68	−0.21
$C =$ 0.50	3.30	0.74	0.92	−1.85	0.8	$C =$ 0.50	3.85	0.07	0.29	−0.41
$D =$ −2.60	4.12	3.83	0.00	0.00	1.0	$D =$ −0.42	4.81	0.90	0.00	0.00

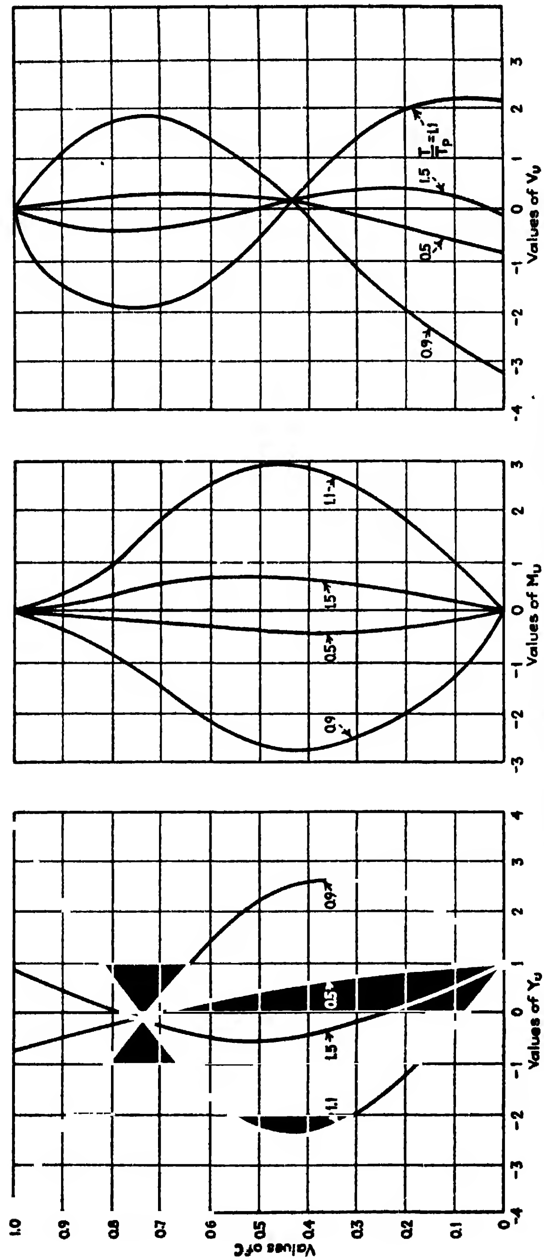


FIG. 12.—Hinged-free beam, forced vibrations.

7.8. Case 4. Forced Vibrations of a Free-free Beam.

For a free-free beam, the conditions, at $x = 0$, are

$$x = 0, \qquad y = F, \qquad \frac{1}{m'^2}\frac{d^2y}{dx^2} = 0$$

It should be noted that the shear, at $x = 0$, can no longer be assumed as zero, since the assigned motion is being accomplished by a periodic force at that point.

The conditions, at $x = L$, are

$$x = L, \qquad \frac{1}{m'^2}\frac{d^2y}{dx^2} = 0, \qquad \frac{1}{m'^3}\frac{d^3y}{dx^3} = 0$$

The end conditions for case 4 are therefore qualitatively the same as for case 3. The solutions are identical. Figure 12, and Eqs. (180), (181), and (182) apply to both cases.

It should be kept in mind, however, that the free periods for the two cases differ. Consequently, for beams identical, except for the end conditions, the deflections, moments, and shears will differ quantitatively.

7.9. Problem.

A vertical steel cantilever, 1 ft. square in section, projects 77 ft. 0 in. above its footing which is set into soft clay. Assuming that the cantilever is free to rotate in a vertical plane, what is its fundamental free period of vibration? What will be the values of the maximum deflections, moments, and shears during a destructive earthquake if the amplitude of the footing is 2.50 in.?

Assuming that the end conditions are hinged-free, the fundamental free period will be given by Eq. (98), or

$$T = 0.41\sqrt{\frac{wL^4}{EIg}}$$

L = 77 ft. 0 in. = 924 in.; $L^4 = 73 \times 10^{10}$.
w = 490 lb. per linear foot = 40.8 lb. per inch.
E = 30,000,000 lb. per square inch.
$I = \frac{1}{12} \times 12 \times 12^3 = 1728$ in.^{4}; $EI = 5184 \times 10^7$.
$g = 32.2 \times 12 = 386$ in. per second per second.

Therefore

$$T = 0.50 \text{ sec.}$$

According to Par. 3.4, periods of destructive earthquakes vary between 1.00 and 1.50 sec. The lower value will be critical and will be assumed. Therefore,

$$T_p = 1.00, \qquad \frac{T}{T_p} = 0.50, \qquad \left(\frac{T}{T_p}\right)^{1/2} = 0.71$$

Entering Fig. 12, the following values of Y_u, M_u, and V_u were obtained:

c	Y_u	M_u	V_u
0.0	1.00	0 00	−0 83
0 2	0.82	−0 32	−0 33
0.4	0.56	-0.38	0 07
0.6	0.20	−0.26	0 30
0.8	−0 26	−0.10	0.28
1.0	−0 73	0 00	0.00

By Eqs. (96) and (138),

$$m'L = 3.93\sqrt{\frac{T}{T_p}} = 3.93 \times 0.71 = 2.79$$

From which,

$$m' = \frac{2.79}{924}, \qquad m'^2 = 9.1 \times 10^{-6}, \qquad m'^3 = 27.5 \times 10^{-9}$$

and, since $F = 2.50$ in.,

$$EIm'^2F = 118 \times 10^4, \qquad EIm'^3F = 3560$$

By Eqs. (180), (181), and (182),

$$y = FY_u, \qquad M = EIm'^2FM_u, \qquad V = EIm'^3FV_u$$

The actual deflections, moments, and shears induced during the earthquake are therefore as follows:

c,	y, in.	Y, in.	M, in.-lb.	V, lb.	V', lb.
0.0	2 50	0 00	0	2960	5920
0.2	2.05	0 45	378,000	1180	2360
0.4	1 40	1 10	449,000	250	500
0.6	0.50	2.00	307,000	1070	2140
0 8	−0.65	3.15	118,000	1000	2000.
1.0	−1.83	4.33	0	0	0

where $c = x/L$

y = deflection relative to the space axis.

$Y = F - y$ = deflection relative to the foundation.

M = dynamic moment.

V = dynamic shear.

$V' = 2V$ = static shear equivalent of dynamic shear.

7.10. References.

DANUSSO, A.: "Statics of Aseismic Construction."

LAMB, H.: "Dynamical Theory of Sound."

LECONTE, J. N., and J. E. YOUNGER: Stresses in a Vertical Elastic Rod when Subjected to a Harmonic Motion at One End, *Bull. Seis. Soc. Amer.*, March, 1932.

PRESCOTT, J.: "Applied Elasticity."

RAYLEIGH, LORD: "Theory of Sound," vol. 1.

Smithsonian Institution, Mathematical Tables, Hyperbolic Functions.

TIMOSHENKO, S.: "Applied Elasticity."

———: "Vibration Problems in Engineering."

CHAPTER 8

DISTRIBUTION OF SEISMIC SHEAR AND MOMENT

8.1. Distribution of Shear.

The mass-acceleration force of an earthquake acts against the mass of a building and will therefore have the greatest effect at the floor levels where most of the weight is concentrated. If the floors are made so stiff that they deflect as units, the deflection of all the columns and walls at any floor level will be identical. Again, if the stiffness factor of a floor is ten or more times that of any column or wall, their ends may be considered as fixed. Under these assumptions, *i.e.*, that the mass of a building is concentrated at the floor levels, that the floors deflect as units, that the ends of the columns and walls are fixed, the columns and walls between two consecutive floors may be treated as *vertical beams*, deflecting equally by reason of a concentrated horizontal load transmitted by the upper floor and with end conditions *fixed-free* but *guided.*

The seismic shear and moment at any floor are taken by the columns and exterior and interior walls in the floor below. However, interior walls are weak structurally and their arrangement is subject to change. Unless they are specifically designed to function, their part in resisting the shear and moment will be neglected.

By Eq. (59), the maximum shear and moment deflection of a fixed-free but guided steel beam with a concentrated load at the free-but-guided end is

$$Y' = \frac{PL(L^2 + 36r^2)}{12E_sI} \tag{183}$$

By Eq. (70), the maximum shear and moment deflection of a similar reinforced-concrete beam is

$$D = \frac{PL(L^2 + 36r^2)}{18E_sIK'} \tag{184}$$

By hypothesis, at the same elevation,

$$Y' = D \tag{185}$$

where Y' = total deflection at the free-but-guided end, for the steel beam.
D = total deflection, as above, for the reinforced-concrete beam.
P = total horizontal seismic force transmitted by the upper floor.
L = clear span of the vertical beams.
r = radius of gyration.
E_s = Young's modulus for steel.
I = moment of inertia of the section parallel to P.
K' = coefficient defined in Par. 5.7.

Let

$$Z = \frac{(L^2 + 36r^2)}{I}$$

for the steel beams, and

$$G = \frac{(L^2 + 36r^2)}{IK'}$$

for the reinforced-concrete beams.

Substituting Z and G for their equivalents in Eqs. (183) and (184), the subscripts s and c denoting steel and concrete, respectively,

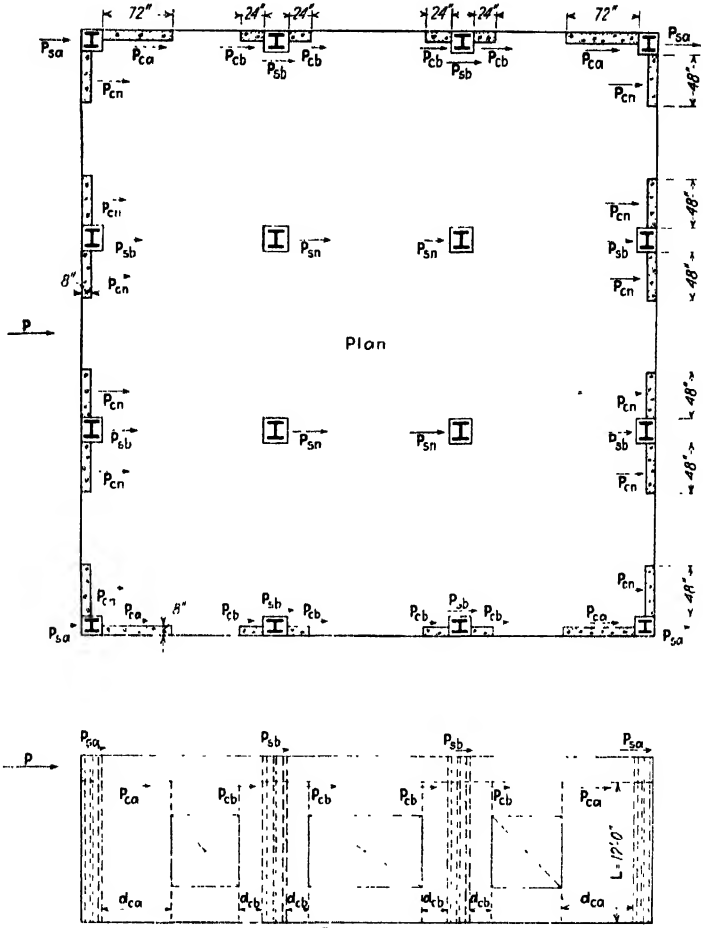
$$Y' = \frac{P_sLZ}{12E_s} \tag{186}$$

$$D = \frac{P_cLG}{18E_s} \tag{187}$$

Referring to Fig. 13,

$$Y' = \frac{P_{sa}LZ_a}{12E_s} = \frac{P_{sb}LZ_b}{12E_s} = \frac{P_{sn}LZ_n}{12E_s} = D \tag{188}$$

$$D = \frac{P_{ca}LG_a}{18E_s} = \frac{P_{cb}LG_b}{18E_s} = \frac{P_{cn}LG_n}{18E_s} = Y' \tag{189}$$



Elevation

FIG. 13.

Therefore,

$$\frac{P_{sa}}{P_{sb}} = \frac{Z_b}{Z_a} = c_1 \tag{190}$$

$$\frac{P_{sa}}{P_{sn}} = \frac{Z_n}{Z_a} = c_2 \tag{191}$$

$$\frac{P_{sa}}{P_{ca}} = \frac{2G_a}{3Z_a} = c_3 \tag{192}$$

$$\frac{P_{sa}}{P_{cb}} = \frac{2G_b}{3Z_a} = c_4 \tag{193}$$

$$\frac{P_{sa}}{P_{cn}} = \frac{2G_n}{3Z_a} = c_5 \tag{194}$$

and

$$P_{sb} = \frac{P_{sa}}{c_1}, \qquad P_{sn} = \frac{P_{sa}}{c_2}, \qquad P_{ca} = \frac{P_{sa}}{c_3}, \qquad P_{cb} = \frac{P_{sa}}{c_4},$$
$$P_{cn} = \frac{P_{sa}}{c_5} \tag{195}$$

Let n_1, n_2, n_3, n_4, n_5, and n_6 equal the number of similar beams. Then,

$$P = n_1P_{sa} + n_2P_{sb} + n_3P_{sn} + n_4P_{ca} + n_5P_{cb} + n_6P_{cn} \tag{196}$$

Substituting their equivalents from Eq. (195) for P_{sb}, P_{sn}, P_{ca}, P_{cb}, and P_{cn}, Eq. (196) becomes

$$P = n_1P_{sa} + \frac{n_2}{c_1}P_{sa} + \frac{n_3}{c_2}P_{sa} + \frac{n_4}{c_3}P_{sa} + \frac{n_5}{c_4}P_{sa} + \frac{n_6}{c_5}P_{sa} \tag{197}$$

from which

$$P_{sa} = \frac{P}{\left(n_1 + \frac{n_2}{c_1} + \frac{n_3}{c_2} + \frac{n_4}{c_3} + \frac{n_5}{c_4} + \frac{n_6}{c_5}\right)} \tag{198}$$

With P_{sa} known, P_{sb}, P_{sn}, P_{ca}, P_{cb}, and P_{cn} are determined by Eq. (195).

8.2. Distribution of Moment.

Since the seismic moment taken by the vertical beams at any floor is proportional to the seismic shear, it follows that

$$M_{sa} = \frac{M}{\left(n_1 + \frac{n_2}{c_1} + \frac{n_3}{c_2} + \frac{n_4}{c_3} + \frac{n_5}{c_4} + \frac{n_6}{c_5}\right)} \tag{199}$$

and

$$M_{sb} = \frac{M_{sa}}{c_1}, \qquad M_{sn} = \frac{M_{sa}}{c_2}, \qquad M_{ca} = \frac{M_{sa}}{c_3},$$
$$M_{cb} = \frac{M_{sa}}{c_4}, \qquad M_{cn} = \frac{M_{sa}}{c_5}. \tag{200}$$

8.3. Shearing, Tensional, and Compressive Stresses.

For the steel beams,

$$v = \frac{P_{sn}}{bd}, \qquad f = \frac{M_{sn}d}{2I} \tag{201}$$

where v = average unit shearing stress.
bd = sectional area = A.
d = depth of beam.
f = unit extreme fiber stress in tension and compression.

For the reinforced-concrete beams,

$$v = \frac{P_{cn}}{jbd}, \qquad u = \frac{P_{cn}}{ojd}, \qquad f_s = \frac{M_{cn}}{pjbd^2}, \qquad f_c = \frac{2M_{cn}}{jkbd^2} \tag{202}$$

where j = ratio of lever arm of resisting couple to d.
k = ratio of depth of neutral axis below the compressive face to d.
u = unit bond stress.
o = summation of the circumferences or perimeters of bars.
b = width of beam.
p = steel ratio = (A_s/bd).
A_s = sectional area of the tensile steel.
f_s = unit stress in the tensile steel.
f_c = unit stress in concrete.

The vertical beams will be designed to take stress reversals, *i.e.*, both faces will be reinforced.

8.4. Maximum Seismic Stresses.

It will be found that the largest unit stresses will occur in the vertical beams which are most rigid in the plane of bending, and that, if the unit stresses in these beams do not exceed the allowable stresses, the same will likewise be true of the stresses in the other vertical beams. It will also be found that the maximum depth of beam which can be used is limited by the allowable bending stresses.

8.5. Problem.

The seismic shear P and moment M transmitted to the vertical beams shown in Fig. 13 are P = 200,000 lb. and M = 14,400,000 in.-lb. What

proportion of the shear and moment is taken by the vertical beams *sa*, *sb*, *sn*, *ca*, *cb*, and *cn*, if

sa = 14 in. B.H. at 100 lb.
sb = 14 in. B.H. at 84 lb.
sn = 14 in. B.H. at 168 lb.
L = 12 ft. 0 in. = 144 in.
$Z = (L^2 + 36r^2)/I$

ca = 8 in. × 72 in.
cb = 8 in. × 24 in.
cn = 48 in. × 6 in.
p = 0.007
$G = (L^2 + 36r^2)/IK'$

Steel beams	n	A	r	r^2	I	Z
sa	4	29.36	3.49	12.2	356.9	59.3
sb	8	24.76	3.45	11.9	294.5	71.8
sn	4	49.51	3.82	14.6	720.6	29.5

By Eq. (190),

$$c_1 = \frac{71.8}{59.3} = 1.211$$

By Eq. (191),

$$c_2 = \frac{29.5}{59.3} = 0.497$$

RC beams	n	b	d	A	I	IK'	r^2	G
ca	4	8	70	560	228,700	21,980	409.0	1.615
cb	8	8	22	176	7,110	683	40.4	32.4
cn	12	48	6	288	864	83	3.0	251.0

K' was determined by Eqs. (62) and (63); $r^2 = I/A$.

By Eq. (192),

$$c_3 = \frac{3.23}{177.9} = 0.01818$$

By Eq. (193),

$$c_4 = \frac{64.8}{177.9} = 0.364$$

By Eq. (194),

$$c_5 = \frac{302}{177.9} = 1.70$$

By Eq. (198),

$$P_{sa} = \frac{200{,}000 \text{ lb.}}{267.75} = 747 \text{ lb.}$$

Therefore, by Eq. (195),

$$P_{sb} = \frac{747}{1.211} = 616 \text{ lb.}$$

$$P_{sn} = \frac{747}{0.497} = 1,501 \text{ lb.}$$

$$P_{ca} = \frac{747}{0.01818} = 41,100 \text{ lb.}$$

$$P_{cb} = \frac{747}{0.364} = 2,050 \text{ lb.}$$

$$P_{cn} = \frac{747}{1.70} = 440 \text{ lb.}$$

In a like manner, by Eq. (199),

$$M_{sa} = \frac{14,400,000 \text{ in.-lb.}}{267.75} = 53,800 \text{ in.-lb.}$$

And, by Eq. (200),

$$M_{sb} = \frac{53,800}{1.211} = 44,400 \text{ in.-lb.}$$

$$M_{sn} = \frac{53,800}{0.497} = 108,100 \text{ in.-lb.}$$

$$M_{ca} = \frac{53,800}{0.01818} = 2,960,000 \text{ in.-lb}$$

$$M_{cb} = \frac{53,800}{0.364} = 148,000 \text{ in.-lb.}$$

$$M_{cn} = \frac{53,800}{1.70} = 31,700 \text{ in.-lb.}$$

Investigating the beam most rigid in the plane of bending, namely, *ca*, and assuming that $j = \frac{7}{8}$, and $k = \frac{3}{8}$, it is found that

$$v = \frac{P_{ca}}{jbd} = \frac{41,100 \text{ lb.}}{\frac{7}{8} \times 8 \times 70} = 84 \text{ lb. per square inch}$$

$$f_s = \frac{M_{ca}}{pjbd^2} = \frac{2,960,000 \text{ in.-lb.}}{0.007 \times \frac{7}{8} \times 8 \times 70^2} = 12,300 \text{ lb. per square inch}$$

$$f_c = \frac{2M_{ca}}{jkbd^2} = \frac{2 \times 2,960,000 \text{ in.-lb.}}{\frac{7}{8} \times \frac{3}{8} \times 8 \times 70^2} = 459 \text{ lb. per square inch}$$

The area of tensional steel required is,

$$A_s = pbd = 0.007 \times 8 \times 70 = 3.92 \text{ sq. in.}$$

Assuming that four one-inch square bars are used, the sum of the perimeters, *o*, is 16 in. Therefore,

$$u = \frac{P_{ca}}{ojd} = \frac{41,100 \text{ lb.}}{16 \times \frac{7}{8} \times 70} = 42 \text{ lb. per square inch}$$

8.6. References.

BROWN, A. R.: Earthquake Design Data, U. S. Veterans Bureau.
NAITO, T.: "Earthquake Resisting Construction."
PORUSH, W.: Earthquake Stresses in Rigid Building Frames, *Bull. Seis. Soc. Amer.*, January, 1933.
SUTHERLAND and CLIFFORD: "Reinforced Concrete Design."
U. S. Navy, Standards of Design for Concrete.

CHAPTER 9

APPLICATION TO BUILDINGS

9.1. Investigation of the Site.

It is possible to study the earth's crust in the same manner as any other structure. Its flaws are found partly by searching for rifts and faults, partly by the differences in elasticity of the component materials, and partly by the wide temperature variations in those regions which are seismically active. In investigating a site for an important structure, the following items should be determined:

a. Proximity of rifts and faults.
b. Dominant periods of vibration.
c. Soil characteristics.
d. Character of the substrata.

9.2. Proximity of Faults.

The proximity to the site of rifts and of active and non-active faults should be ascertained. Seismic data about any particular locality may be obtained from near-by seismological stations and from the U. S. Coast and Geodetic Survey or may be determined independently by a series of seismographic observations. It is recommended that an experienced structural geologist be consulted about the location of rifts and concerning the soil characteristics of the site.

It would be unwise, even in the absence of all signs of seismic activity, to locate a building over a known fault or rift, for it is highly probable that in the event of a slip of any magnitude the building will be destroyed.

9.3. Dominant Periods of the Site.

The site may vibrate in one or more periods depending on its composition and dimensions. If it consists of dry,

well-consolidated, highly elastic material, it has usually only a predominant period. If, however, it is composed of marshy or semifluid matter, it may have several dominant periods. The Suyehiro vibration analyzer, or the experimental seismograph devised by Neumann of the U. S. Coast and Geodetic Survey, may be used to determine these periods, which are required in order to be able to assign probable ground amplitudes during destructive earthquakes. The closer the periods approach those in the dangerous bracket of periods, or those of the proposed building, the larger the amplitudes assumed.

9.4. Soil Characteristics.

The nature of the materials constituting the site, their elasticity, moisture content, and uniformity determine the damping value of the site, its probable amplitude during a destructive earthquake, and the character of foundation required.

Thus, if the material is bedrock whose elastic value is high, the damping value is poor and the amplitude assigned is low.

Again, if the site is composed of dry, compact, uniform alluvial or diluvial deposits whose elastic value is less than that of bedrock, the damping value is better and the amplitude assigned is larger. Where the materials contain an appreciable amount of moisture, the elastic value becomes low, the damping is excellent, and the amplitude assumed is high, its value depending on the cohesive strength of the soil and on how closely it approaches the condition of semifluidity.

Finally, if the soil is not uniform but consists of large adjoining deposits of varying physical properties, the damping value is good and the amplitude assumed is an average value depending on the nature of the component materials. It is evident, however, that each of these materials will have its own amplitude so that a differential movement of the site may be expected during an earthquake.

9.5. Character of the Substrata.

Borings should be driven to a depth of at least twenty feet below the bottom of the footings, so as to determine the nature of the substrata and the ground-water level. If the borings indicate that the substrata are less elastic and have a greater moisture content than the overlying material, the amplitude assigned must reflect the fact that the greater amplitude of a more mobile soil is not prevented by a superincumbent layer of drier and more compact material, the overlying strata being displaced in the same general order as the more mobile material below.

If the borings show that the ground-water level is higher than three feet below the elevation of the bottom of footings, an adequate system of subsurface drainage should be provided to lower the ground-water level to the required elevation.

9.6. Principle of Discontinuity.

During an earthquake, the soil forces the foundation of a building through a definite amplitude. The most effective way of decreasing that amplitude is by insulating the foundation from the native soil. This can be done, to a certain extent, by eliminating any condition of fixity between the foundation and the soil, *i.e.*, by creating *discontinuity.* Both theory and practice show that discontinuity causes a pronounced decrease in the amount of amplitude transmitted. The principle involved is simple, being nothing more than creating inefficiency in the transmission of a force. This is comparable to what happens when the fan belt of an engine is loosened; the fan slows up although the engine still rotates at the same number of revolutions per minute.

A 3-ft. layer of gravel or broken stone introduced between the foundation and the native soil is deep enough to create discontinuity. It is advantageous to use gravel rather than sand, because it is a more stable material, is less likely to shift and settle or to absorb water which transmits vibrations readily. Based on experimental results in

vibration engineering, it is estimated that a 3-ft. bed of gravel or broken stone will decrease the amplitude transmitted to the foundation by one half.

9.7. Foundations.

Figure 2 shows that locating foundations at a depth greater than is required ordinarily, does not materially decrease the amplitude transmitted. Thus, at a subsurface depth of 50 ft., the amplitude is only 10 per cent less than at the surface.

A correctly designed foundation is a vital factor in the ability of a structure to weather a major earthquake undamaged. The ideal aseismic foundation is a beam-and-slab mat. This not only cancels the adverse effect of differential soil movements and does away with the possibility of footings digging themselves in and destroying the desired state of discontinuity but also lends itself to a rigid and monolithic design. To be most effective, the beams and slab should be poured integrally.

If a mat is out of the question, and individual footings are used, the elevations of the bottoms of the footings should all be identical and the footings should be interconnected by beams able to maintain them in that same relative position. Owing to the oscillation of the center of gravity of a building during vibration, the exterior parts of the foundations should be designed to withstand soil pressures as high as four times the static soil pressure.

On bedrock or on dry, well-compacted alluvial or diluvial deposits which yield uniformly and moderately under pressure, a system of interconnected footings may be used. If, however, the soil is not uniform, yields excessively under pressure, or contains considerable moisture, a beam-and-slab foundation is indicated.

Where the site consists of wet, loose fill, or approaches a state of semifluidity, a beam-and-slab foundation on piles is required. Where soil values permit, short piles, which allow the building to rotate in a vertical plane, should be used rather than long piles driven to hardpan.

9.8. Buildings.

The fundamentals of satisfactory earthquake-proof design are: symmetry of plan, mass, and rigidity. Buildings should have a square or rectangular plan, these types having proved strongest in resisting earthquakes. The center of mass should coincide with the center of rigidity. Both mass and rigidity should be as nearly uniform throughout the building as possible, so that the building cannot rotate about a vertical axis owing to one part being heavier or stiffer than another. Rigid bents should be symmetrical with respect to the floor plan. All parts of the building should be so connected to the frame that the building will vibrate as a unit, the connections being strong enough to overcome the inertia of the separate parts. Buildings should be fireproof and as light as the required loading permits. They should be protected from fire by an earthquake-proof sprinkler system. Where adjacent buildings have different periods or amplitudes, they should be separated by an amount sufficient to keep them from hammering one another during an earthquake.

In tied columns, the ties should be spaced closely for some distance above and below the floors. Diagonal bands of steel should extend back from the corner columns into the floor slab. Provide additional end restraint for columns by means of extra bars in the floor slab and across the columns.

The floors act as horizontal girders and should be designed as such. The floor system should be so braced diagonally that it will deflect as a unit and will be strong enough to transmit the lateral forces to all the bents. The full depth of the wall spandrels should be designed as beams, so as to attain the maximum possible floor rigidity. Floor slabs should be of rock concrete, rigid and monolithic. Horizontal diagonal bars shall be used between floor beams connecting interior columns.

Exterior walls should be of reinforced concrete, of uniform thickness, and designed as vertical beams. Use ties and

diagonal bars. All bars shall be hooked where possible. Diagonal bars shall be used across all wall openings. Windows or other wall openings shall be kept away from the corner columns. All construction joints shall be doweled, as this increases their reliability. Fasten partitions to the frame. Wall bents should be continuous for the full height of the building. Temperature steel should be placed diagonally. Stagger reinforcing bar joints.

Fasten fire towers to the building frame and not to filler walls. Brick or stone veneer or other ornamentation should be tied to the frame by adequate ties. Chimneys should be of reinforced concrete or of sheet metal. Use 1:3 Portland cement mortar. Use metal lath partitions. Make all piping and conduits of lead. Lay mains in a trench beneath the building, so that no piping is set into the building. Let risers hang free, with bends through sleeves to the fixtures. If this is done, no piping or wiring will be injured during an earthquake.

9.9. End Conditions.

In order to apply the findings of Chaps. 6 and 7 to buildings, certain arbitrary assumptions will now be made. Thus, the end conditions at the top of a building will be considered as *free*. The end condition at the bottom of a building is determined by the nature of the soil, the type of foundation, and the state of discontinuity.

For example, if the footings are fixed into bedrock, or into dry, compact, alluvial or diluvial deposits, the end condition will be assumed as *fixed*. Where a layer of gravel separates the foundation from the native soil, and the construction is designed to permit the foundation to deflect laterally, the end condition will be considered as *sliding*.

Again, if the foundation is fixed into wet, compressible deposits of such nature that rotation in a vertical plane can occur, the end condition will be assumed as *hinged*. Where a layer of gravel separates the foundation from the soil, and the building is free to move in an horizontal as well as in a

vertical plane, the end condition will be considered as approximately *free.*

It is obvious that these end conditions have no definite limits but merge into one another. Their determination for any particular case may be further complicated by reason of the lateral constraint exercised by contiguous buildings. After all the conditions involved are evaluated, therefore, it becomes a matter of engineering judgment to decide how closely any of the arbitrary standards are approached and what value of c shall be assumed. The values of c given are the numerical coefficients of the fundamental-period equations (89), (98), (107), and (116).

VALUES OF c (FUNDAMENTAL)

Free-free	Hinged-free	Sliding-free	Fixed-free
0.28	0.41	1 12	1.79

9.10. Periods of Buildings.

Assuming that the vibration of a tall structure, *i.e.*, one where the ratio L/d is greater than 5, is similar to that of a slender beam—and both theoretical considerations and examination of the vibration records of tall buildings indicate that this assumption is probably correct—the period of vibration of tall buildings may be expressed as

$$T = c\sqrt{\frac{wL^4}{EIg}}$$

where T = free period, seconds.
c = a constant depending on the end conditions.
w = weight, pounds per inch of L.
L = height of building, inches.
E = Young's modulus, pounds per square inch.
I = moment of inertia of the section perpendicular to the plane of vibration, inch^4.
g = acceleration due to gravity, inches per second per second.

It is essential that the value assigned to c represents actual conditions, since it is very influential in calculating the period.

For short buildings, *i.e.*, where the ratio L/d is less than 5, the value of T obtained should be modified by Eq. (129), which states that

$$T_t = \sqrt{T^2\left(1 + \frac{d^2}{4c_1L^2}\right)}$$

where T_t is the true period of a short building, and c_1 is a deflection constant whose values are given below. The values of c_1 are arbitrary, being based partly on the assumptions of Par. 9.9.

VALUES OF c_1

Fixed-free	Sliding-free	Hinged-free	Free-free
0.13	0.20	0.55	0.80

The weight per linear inch w is an average value. It is obtained by adding to the dead weight, a live load per square foot of floor area, depending on the actual conditions of usage, and dividing the total by the height of the building L. For preliminary estimates, the following values of w may be used for buildings where the live load does not exceed 100 lb. per square foot:

w = 1.2 A' lb. per inch for buildings less than 100 ft. high.
w = 1.5 A' lb. per inch for buildings taller than 100 ft.
A' = gross area of the average section, square feet.

For prismatical buildings, L is taken as the distance from the top of roof slab to the point of fixity. Where the foundation is not fixed, L is the distance from the top of roof slab to the bottom of the foundation. If the building is not prismatical, an equivalent height is determined.

The quantity EI is taken as the average value along L. It will be calculated by the formula

$$EI = \frac{a_1(EI)_1 + a_2(EI)_2 + a_3(EI)_3}{L}.$$

where $(EI)_1$ = EI of section parallel to and above the average floor.

$(EI)_2$ = EI of section through the average equivalent floor.

$(EI)_3$ = EI of section through the foundation.

And

a_1, a_2, and a_3 = lineal extent of $(EI)_1$, $(EI)_2$, and $(EI)_3$, respectively.

As used above, the term "average equivalent floor" means the theoretical depth of floor slab over the gross area of the plan obtained by considering the transformed volumes of the actual floor skeleton and slab at the average floor.

9.10*A*. Periods of Earthquakes.

The curves shown in Figs. 9, 10, 11, and 12 illustrate the profound influence exerted by the ratio T/T_p on the theoretical seismic moments and shears. Obviously, the assumption of the correct earthquake period T_p is vital in determining the theoretical stresses to be resisted by a structure. Although, as stated in Par. 3.4, considerable opinion points to the fact that destructive earthquakes occur in periods ranging from 1.0 and 1.5 sec., there is nevertheless no assurance that any assumed T_p will be even approximately correct.

However, the gravest stresses will occur as T_p approaches T, *i.e.*, as the period of the earthquake synchronizes with that of the structure. The author believes that the worst case which need be considered practically will be covered by assuming a ratio of T/T_p of 0.9 or of 1.1. His reasoning is as follows: That complete synchronism is highly improbable, but, if it should occur, the inherent factor of safety of the structure will be ample to take care of the temporary increased stresses provided that the structure has been designed for a T/T_p of 0.9 or of 1.1.

Therefore, although for intermediate structures the method of arriving at the ratio T/T_p by comparing T with the dangerous bracket of periods may suffice, nevertheless,

for very important structures or for maximum safety, it is recommended that a ratio of T/T_p of 0.9 or of 1.1 be adopted.

9.11. Measurements of Periods.

The apparatus shown in Fig. 14 may be used to measure and record the periods of vibration of buildings, provided that the periods are not too small, *i.e.*, less than 0.3 sec. In the latter case, an instrument such as the Neumann experimental seismograph will yield more accurate results.

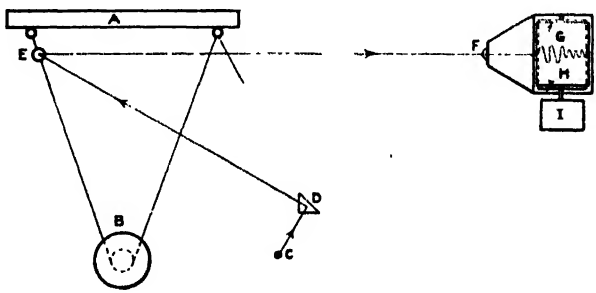


FIG.—14. Vibrograph.

The apparatus consists of a frame A, from which is suspended a bob B by a string adjustable as to length. This permits the period of the pendulum to be varied at will. As the period of the pendulum approaches one of the critical periods of the building, the amplitude of the motion of the bob will become relatively large owing to synchronism. The period of the pendulum, and consequently of the building, may then be calculated by the following formula:

$$T = 0.55(L)^{\frac{1}{2}}$$

where T is the period of the pendulum in seconds, and L is the length of the pendulum in feet.

To record the vibrations, a concentrated source of light C is focused by prism D on mirror E, which reflects a spot of light to lens F, focusing it on photographic paper G, which covers drum H, driven at constant speed by clockwork I. The critical periods of the building are determined

from the gram by measuring those recorded vibrations which show abnormal amplitudes.

Measuring the building periods both before and after an earthquake affords a powerful and direct method of checking against invisible seismic damage. Thus, if the fundamental period of a building has increased, consideration of the general period equation

$$T = c\sqrt{\frac{wL^4}{EI}}$$

discloses that only the terms c or EI or some combination of both could have changed. It is unlikely that c, which represents the end conditions, has changed. At any rate, this point could be determined by comparing the new T/T_1 with the old by the method of Par. 6.8. The more probable occurrence is that the term EI, which represents the rigidity, has decreased. The author is of the opinion that an increased building period after an earthquake is an almost infallible sign of rupture somewhere within the frame, necessitating investigation and repair.

9.12. Longitudinal Periods.

The longitudinal periods of tall buildings are less than the transverse periods. Consequently, except when the longitudinal periods fall in the dangerous bracket, the ratio T'/T_p and the vertical seismic moments and shears will be less than the ratio T/T_p and the horizontal seismic moments and shears. Furthermore, buildings are primarily designed to take vertical loads, and their factor of safety can readily take care of the increased vertical loading due to earthquakes, unless the vertical period falls in the dangerous bracket of periods. In that event the vertical seismic moments and shears should be investigated.

9.13. Design Procedure.

A complete investigation of the earthquake resistant design of a building includes the following steps:

1. A seismological study of the site and locality with particular reference to its earthquake history, dominant periods of vibration, and proximity to faults and rifts.

2. A geological survey of the site to determine the nature of the soil, the depth of the strata, and the elevation of the ground-water level.

3. Selection of the type of foundation to be used.

4. Decision as to the probable end conditions of the building. Selection of c.

5. Determination of the foundation amplitude.

6. Design of the structural frame for the statical loads, *i.e.*, dead, live and wind loads, making the floor systems as deep and rigid as possible.

7. Symmetrical distribution of the vertical reinforced-concrete beams.

8. Calculation of the transverse fundamental periods of the building about the X-X and Y-Y axes.

9. Calculation or determination of the ratios T/T_p. If T falls in the dangerous bracket, assume that $T/T_p = 0.9$ or 1.1.

10. Determine the maximum deflections, moments, and shears by means of Figs. 9, 10, 11, and 12, and Eqs. (161), (162), (163), (180), (181), and (182).

11. For very important structures, check these values by tests of models according to the method of Par. 9.14.

12. Distribute the seismic shears and moments among the vertical beams. Check the stresses in the vertical beams.

13. Design, detail, and construct the building in conformity with the principles outlined in Par. 9.8.

14. Check the column sections for the combined stresses due to the statical and seismic loadings.

15. Measure the periods of the completed structure and check on the assumed end conditions, computed periods, ratios T/T_p, deflections, moments, and shears.

9.14. Tests of Models.

A graphic check on the theoretical seismic deflections of a building can be obtained by means of the vibration

of models on a *shaking platform.* The materials constituting the models must be of such nature that the amplitudes will be neither too large nor too small. The shaking platform is forced through the harmonic motion of the desired period and amplitude by means of a cam incorporated in the shaking mechanism. Moving pictures are taken during the experiments, and the deflections of the model are measured on the projected film. The deflections of the model will bear a definite relationship to those of the building provided that its material and loading are such as to satisfy the principle of *dynamic similitude, i.e.,* that

$$\frac{U}{E} = \frac{U'}{E'}, \qquad \frac{Ld}{E} = \frac{L'd'}{E'}, \qquad \frac{M}{EL^2} = \frac{M'}{E'L'^2}, \qquad \frac{F}{L} = \frac{F'}{L'}$$

$$Lp^2 = L'p'^2$$

where U = ultimate strength of the building.
E = Young's modulus of the building material.
L = linear dimension of the building.
d = density of the building material.
M = mass loaded on the building.
F = forced amplitude of the foundation of the building.
p = forced angular velocity.
g = acceleration due to gravity.

The accented letters refer to the model.

9.15. References.

ALVAREZ, A. C.: University of California Engineering Publications.

American Concrete Institute, Proceedings of.

American Society of Civil Engineers, Proceedings of.

Beton und Eisen, Berlin.

BIOT, M.: *Proc. Nat. Acad. Sci.*, February, 1933.

BUTTS, W. M.: Publications of the Clay Products Institute of California.

BYERLY, P., J. HESTER, and K. MARSHALL: Natural Periods of Vibration of Tall Buildings in San Francisco, *Bull. Seis. Soc. Amer.*, December, 1931.

COYLE, D. C.: Coyle Vibration Recorder, *Eng. News-Record*, vol. 106, p. 310.

DEBUSSY, M.: Effect of Earthquake Shocks on Structures. *Proc. Amer. Soc. Civil Eng.*, 1928 and 1929.

DEWELL, H. D., and B. WILLIS: Earthquake Damage to Buildings, *Bull. Seis. Soc. Amer.*, December, 1925.
Earthquake Research Institute of Japan, Bulletin of.
Engineering News-Record, Files of.
ESTES, L. A.: "Earthquake Proof Construction."
FLEMING, R.: "Wind Stresses in Buildings."
Fourth Pan-Pacific Science Congress, Proceedings of.
FREEMAN, J. R.: "Earthquake Damage and Earthquake Insurance."
HALL, E. E.: Vibrations of Buildings Due to Street Traffic, *Eng. News*, Aug. 1, 1912.
HOFFMAN, F. L.: "Earthquake Hazards and Earthquake Insurance."
Illinois, University of, Tests on Concrete Frames, *Eng. News-Record*, July 10, 1930.
Imperial Academy of Japan, Proceedings of.
Iowa State College, Physical Properties of Earths.
JACOBSEN, L. S.: Shaking Table Experiments, *Bull. Seis. Soc. Amer.*, September, 1930.
KIMBALL, A. L.: "Vibration Prevention in Engineering."
MARTEL, R. R.: *Bull. Seis. Soc. Amer.*, September, 1929.
MONTEL, A.: "Structures in Earthquake Countries."
NAITO, T.: "Earthquake Resisting Construction."
OMORI, F.: Imperial Earthquake Investigation Committee, *Publication* 20, p. 73; *Jour. Inst. Japanese Architects*, vol. 41., p. 498.
SAITA, T.: Earthquake Proof Construction in Japan, *Proc. Third Pan-Pacific Sci. Cong.*, vol. 2, Sec. D; *Jour. Inst. Japanese Architects*, vols. 39 and 40.
SANO, R.: Notes on Earthquake Proof Construction. *Repts. Imp. Earthquake Investigation Comm.*, vol. 83, *A* and *B*.
SLOCUM, S. E.: Experimental Research on Vibration Insulators, *Trans. Amer. Soc. Civil Eng.*, 1931.
SPURR, H. V.: "Wind Bracing."
SUTHERLAND and CLIFFORD: "Reinforced-Concrete Design."
SUYEHIRO, K., and K. SATO: A Dynamical Model for Investigating the Collapse of Wooden Houses in an Earthquake, *Proc. Imp. Acad. Japan*, vol 6, No. 7.
TANABE, H.: Effects of the Systems of Reinforcement upon the Elasticity and Strength of Concrete Slabs in Shear, *Proc., Fourth Pan-Pacific Sci. Cong.*
Third Pan-Pacific Congress, Proceedings of.
WEISKOPF, S. C.: Mitsui Bank Building, *Eng. News-Record*, June 23, 1927.
WRIGHT, F. L.: Why the Japanese Earthquake Did not Destroy the Hotel Imperial, *Liberty*, Dec. 3, 1927.
YOUNG, L. H.: Portable Apparatus for Measuring Building Vibrations, *Bull. Seis. Soc. Amer.*, September, 1927.
Zeitschrift für Technische Physik, Leipzig.

CHAPTER 10

ASEISMIC DESIGN OF A TALL BUILDING

10.1. Problem.

An earthquake-proof office building is to be erected on a rectangular lot whose dimensions limit the plan of the building to 75 ft. 4 in. by 111 ft. 4 in. Three sides of the lot bind on streets, and the fourth is 10 ft. 0 in. distant from a neighboring office building. The proposed structure will contain 45 floors and 2 basements. The story height is 12 ft. 0 in., and the distance between the top of any floor slab and the bottom of the spandrel beam is 2 ft. 6 in. Taking into account the transformed volumes of the floor skeleton and slab, it is estimated that the equivalent depth of the average theoretical floor slab is 8 in. over the gross area of the plan. The lineal extent of the theoretical floor slab along the height of the building is

$$(1 + 45 + 1)0.67 = 31.4 \text{ ft.}$$

Assuming that the foundation slab is 6 ft. thick, the total height of the building, L, is 570 ft. The lineal extent of the vertical beams, *i.e.*, the columns and bracing walls, equals

$$570 \text{ ft.} - 31.4 \text{ ft.} - 6 \text{ ft. or } 532.6 \text{ ft.}$$

The ratios L/d are

$$\frac{570}{108.7} = 5.2 \text{ and } \frac{570}{72.7} = 7.8.$$

According to Par. 9.10, therefore, this building is classed as tall.

10.2. Seismological and Geological Investigation.

Investigation of the site along the lines suggested in Chap. 9 yielded the following data:

a. The locality is subject to considerable earthquake activity of moderate intensity.

b. There are no faults or rifts in the immediate neighborhood. There is an active fault 10 miles distant.

c. The dominant period of vibration of the locality is 0.1 sec.

d. Four test borings driven to a depth of 50 ft. disclosed bedrock at an average depth of 32 ft. 9 in. below the top of street curb. The overlying deposits consisted of hard clay and firm, coarse, dry sand.

e. No ground water was encountered.

f. The free periods of the neighboring office building were measured. The building is rectangular in plan, extends 84 ft. along one street and 206 ft. along the other, and its height above curb is 206 ft. Its footings are fixed into bedrock. The periods observed were:

Parallel to the 84-ft. side,

$$T = 1.4 \text{ sec.,} \qquad T_1 = 0.2 \text{ sec.}$$

$$\frac{T}{T_1} = 7.0$$

Parallel to the 206-ft. side,

$$T = 1.1 \text{ sec.,} \qquad T_1 = 0.2 \text{ sec.}$$

$$\frac{T}{T_1} = 5.6$$

The average value of T/T_1 is 6.3. Referring to Par. 6.8, it is evident that the average end conditions of this building are approximately *fixed-free*.

10.3. Foundation Amplitude.

In order to reduce the amplitude transmitted to the foundation, and to introduce large frictional forces which will dissipate free vibrations quickly, a 3-ft. layer of gravel will be deposited between the foundation of the proposed building and the bedrock. No special steps will be taken, however, to enable the building to slide freely. For free vibration, therefore, the end conditions of the building will be between *fixed-free* and *sliding-free*. It is decided

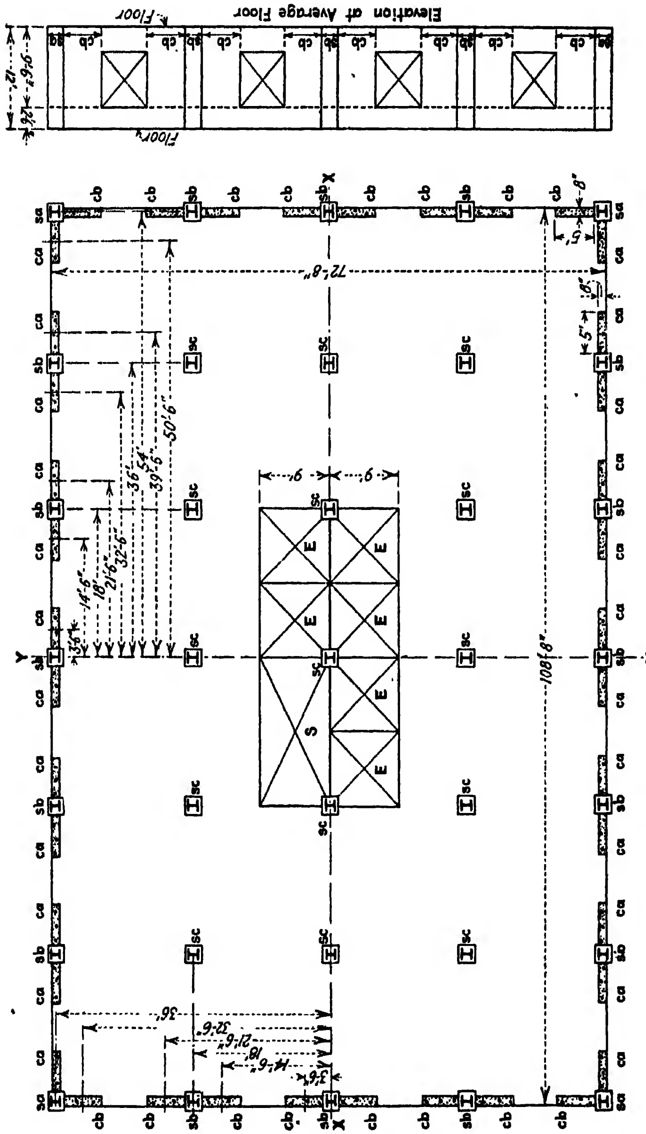


FIG. 15.—Section through vertical beams at average floor.

that the end conditions will be expressed by a value of $c = 1.50$.

By Par. 3.9, the maximum surface amplitude of bedrock, excluding the case of synchronism, is ¼ in. By Fig. 3, the amplitude 33 ft. below the surface, which corresponds approximately to the subsurface depth of the bottom of the gravel bed, is, 0.94 × ¼ in. = 0.24 in. Finally, by Par. 9.6, the amplitude transmitted to the foundation is $F = \frac{1}{2} \times 0.24$ in. = 0.12 in.

10.4. Investigation of the Structural Frame.

The structural frame consists of a steel skeleton with reinforced-concrete roof, floor, and foundation slabs. The frame was designed to carry the dead, live, and wind loads, under the requirements of the local building code and in accordance with the provisions of Chap. 9. A series of reinforced-concrete bracing walls were then assumed and distributed symmetrically about the plan. Figure 15 shows the horizontal cross section over the average floor and an elevation on the average floor. It is assumed that Young's moduli for steel and reinforced concrete have the following average values.

$$E_s = 3 \times 10^7, \qquad E_c = 2 \times 10^6$$

The following table gives the coefficients of the vertical beams shown in Fig. 15, namely, *sa*, *sb*, *sc*, *ca*, and *cb*:

Beam	Section		$I_{x'}$	$I_{y'}$		
sa	16 in. B.H. at 256 lb.	75 4	3,267	1,264	6 58	4 10
sb	16 in. B.H. at 384 lb.	113 1	5,554	2,065	7 01	4 27
sc	16 in. B.H. at 342 lb.	100 6	4,754	1,790	6 87	4 22
ca	R.C., 60 in. × 8 in.	480 0	2,560	144,000	2 31	17 32
cb	R.C., 8 in. × 60 in.	480 0	144,000	2,560	17 32	2 31

10.5X. Analysis about the X-X Axis.

From the theory of the moments of inertia of plane areas,

$$I_x = I_{x'} + Ad_x^2$$

where I_x = moment of inertia of a section about the X-X axis of the building.

$I_{x'}$ = moment of inertia of a section about its own x-x axis.

A = area of the section.

d_x = distance from the center of gravity of the section to the X-X axis of the building.

Let

$(EI)_{x_1}$ = EI of the average section taken above the floor.

I_{xs} = moment of inertia of the steel sections about the X-X axis.

Then

$$
\begin{aligned}
I_{xs} &= 4[3267 + 75.4(36 \times 12)^2] \\
&\quad + 10[5554 + 113.1(36 \times 12)^2] \\
&\quad + 4[5554 + 113.1(18 \times 12)^2] + 2(5554) \\
&\quad + 10[4754 + 100.6(18 \times 12)^2] + 5(4754) \\
&= 3371 \times 10^5
\end{aligned}
$$

and

$$E_s I_{xs} = 10{,}113 \times 10^{12}$$

Let

I_{x_1c} = moment of inertia of the reinforced-concrete sections about the X-X axis.

Then

$$
\begin{aligned}
I_{x_1c} &= 24[2560 + 480(36 \times 12)^2] \\
&\quad + 4[144{,}000 + 480(32.5 \times 12)^2] \\
&\quad + 4[144{,}000 + 480(21.5 \times 12)^2] \\
&\quad + 4[144{,}000 + 480(14.5 \times 12)^2] \\
&\quad + 4[144{,}000 + 480(3.5 \times 12)^2] \\
&= 26{,}438 \times 10^5
\end{aligned}
$$

and

$$E_c I_{x_1c} = 5288 \times 10^{12}$$

But

$$(EI)_{x_1} = E_s I_{xs} + E_c I_{x_1c}$$

Therefore,

$$(EI)_{x_1} = 154 \times 10^{14}$$

Let

$(EI)_{x_2}$ = EI of the average section taken through the equivalent floor slab.

I_{x_2c} = moment of inertia of the equivalent floor slab section about the X-X axis.

Then

$$I_{x_2c} = \tfrac{1}{12} \times 108.67 \times 12(72.67 \times 12)^3 = 721 \times 10^8$$

and

$$E_c I_{x_2c} = 1442 \times 10^{14}$$

But

$$(EI)_{x_2} = E_s I_{xs} + E_c I_{x_2c}$$

Therefore,

$$(EI)_{x_2} = 1543 \times 10^{14}$$

The average value of $(EI)_x$ will therefore be

$$(EI)_x = \frac{532.6(EI)_{x_1} + 31.4(EI)_{x_2} + 6E_c I_{x_2c}}{570}$$

or

$$(EI)_x = 244 \times 10^{14}$$

10.6X. Transverse Period about the X-X Axis.

The transverse period of vibration about the X-X axis is expressed by

$$T_x = c\sqrt{\frac{wL^4}{(EI)_x g}}$$

where T_x = free fundamental period, seconds.
c = 1.50.
w = weight concentration, pounds per inch of height. The dead weight plus a live-load allowance of 20 lb. per square foot of floor area under actual conditions of usage is estimated at 82,100,000 lb. Therefore, w = 12,000 lb. per inch.
$(EI)_x = 244 \times 10^{14}$.
$L^4 = (570 \times 12)^4 = 2189 \times 10^{12}$.
g = 386 in. per second per second.

Therefore,

$$= 1.50\sqrt{\frac{12{,}000 \times 2189 \times 10^{12}}{244 \times 10^{14} \times 386}} = 2.51 \text{ sec.}$$

Assuming a destructive earthquake of period $T_p = 1.50$ sec.,

$$\frac{T_x}{T_p} = \frac{2.51}{1.50} = 1.67$$

and

$$\left(\frac{T_x}{T_p}\right)^{1/2} = 1.29$$

By Par. 6.8, the ratio $T/T_1 = 5.8$ when $c = 1.50$. Therefore, the first harmonic period T_1 about the X-X axis will be

$$T_1 = \frac{2.51}{5.8} = 0.43 \text{ sec.}$$

10.7X. Transverse Deflections, Moments, and Shears during Forced Vibration about the X-X Axis.

During forced vibration, as demonstrated in Par. 7.5, the end conditions change qualitatively to *fixed-free*. Therefore, entering Figs. 9, 10, and 11, and interpolating roughly between the curves marked $T/T_p = 1.5$ and $T/T_p = 3.0$, the following values of Y_u, M_u, and V_u are obtained:

c	Y_u	M_u	V_u
0.0	1.00	−1.30	0.05
0.2	0.85	−1.25	0.50
0.4	0.45	−0.95	0.80
0.6	−0.20	−0.55	0.90
0.8	−0.95	−0.15	0.65
1.0	−1.75	0.00	0.00

By Eqs. (114) and (138),

$$m'L = 1.88\left(\frac{T_x}{T_p}\right)^{1/2} = 1.88 \times 1.29 = 2.42$$

$$m' = \frac{2.42}{6840} = 3.54 \times 10^{-4}$$

$$m'^2 = 12.55 \times 10^{-8}$$

$$m' = 44.4 \times 10^{-12}$$

$$(EI)_x m'^2 F = 244 \times 10^{14} \times 12.55 \times 10^{-8} \times 0.12 = 368 \times 10^6$$

$$(EI)_x m'^3 F = 244 \times 10^{14} \times 44.4 \times 10^{-12} \times 0.12 = 130 \times 10^3$$

Therefore, by Eqs. (161), (162), and (163), the actual deflections, moments, and shears are:

(1) c	(2) y, in.	(3) Y, in.	(4) M, in.-lb.	(5) V, lb.	(6) V', lb.	(7) M', in.-lb.
0.00	0.12	0.00	-479×10^6	6.5×10^3	13×10^3	74×10^4
0.20	0.10	0.02	-460×10^6	65×10^3	130×10^3	741×10^4
0.40	0.05	0.07	-350×10^6	104×10^3	208×10^3	1185×10^4
0.60	-0.02	0.14	-202×10^6	117×10^3	234×10^3	1333×10^4
0.80	-0.11	0.23	-55×10^6	84.5×10^3	169×10^3	964×10^4
1.00	-0.21	0.33	0	0	0	0

where $c = x/L$

$y = FY_u$, or the deflection relative to the space axis.

$Y = F - y$, or the deflection relative to the foundation.

$M = (EI)_x m'^2 F M_u$, or the dynamic moment.

$V = (EI)_x m'^3 F V_u$, or the dynamic shear.

$V' = 2V$, or the static equivalent of the dynamic shear.

M' = moment taken by the vertical beams at any floor. The unsupported length of the vertical beam is 9.5 ft. The lever arm is 9.5 ft./2 = 57 in. Therefore, $M' = 57V'$.

Column 3 shows that the maximum amplitude is at the top of the building and equals 0.33 in. This figure is well within the allowable amplitude of $0.001L = 6.84$ in. Column 6 shows that the maximum shear occurs at approximately six-tenths the height and equals 234,000 lb., and, by Column 7, the maximum moment taken by any set of vertical beams is 13,330,000 in.-lb.

10.8*X*. Distribution of the Seismic Shear and Moment.

Investigating the average section shown in Fig. 15, by the methods of Chap. 8,

Steel beams	n	A	$r_{x'}$	$r_{x'}^2$	$I_{x'}$	Z
sa	4	75.4	6.58	43 3	3267	4.46
sb	16	113 1	7 01	49.1	5554	2.66
sc	15	100 6	6 87	47.2	4754	3 09

$$Z = \frac{(L_b^2 + 36r_x{}^2)}{I_{x'}}, \qquad L_b = 9.5 \text{ ft.} = 114 \text{ in.},$$

$$L_b^2 = 12{,}996$$

By Eq. (190),

$$c_1 = \frac{2.66}{4.46} = 0.60$$

By Eq. (191),

$$c_2 = \frac{3.09}{4.46} = 0.69$$

Assuming that $p = 0.007$, then, by Eqs. (62) and (63), $K' = 0.097$, and,

Reinforced-concrete beams	n	b	d	A	$I_{x'}$	IK'	$r_{x'}^2$	G
ca	24	60	6	360	1,080	105	3	124.7
cb	16	8	58	464	129,800	12,600	280	1 834

$$G = \frac{(L_b^2 + 36r_{x'}{}^2)}{I_{x'}K'}$$

By Eq. (192),

$$c_3 = \frac{249.4}{13.38} = 18.62$$

By Eq. (193),

$$c_4 = \frac{3.67}{13.38} = 0.274$$

and, by Eq. (198),

$$P_{sa} = \frac{234{,}000 \text{ lb.}}{112.1} = 2080 \text{ lb}$$

By Eq. (195),

$P_{sb} = 3480$ lb., $\quad P_{sc} = 3020$ lb., $\quad P_{ca} = 110$ lb.,

$P_{cb} = 7620$ lb.

Similarly, by Eq. (199),

$$M_{sa} = \frac{13{,}330{,}000 \text{ in.-lb.}}{112.1} = 119{,}000 \text{ in.-lb.}$$

By Eq. (200),

$$M_{sb} = 198{,}000 \text{ in.-lb.}, \qquad M_{sc} = 173{,}000 \text{ in.-lb.},$$
$$M_{ca} = 6{,}400 \text{ in.-lb.}, \qquad M_{cb} = 435{,}000 \text{ in.-lb.}$$

10.9X. Shearing, Tensional, and Compressive Stresses.

Investigating the beam most rigid in the plane of bending, namely, cb, it is found, by Eq. (202), assuming that $j = \frac{7}{8}$ and $k = \frac{3}{8}$, that

$$v = 19 \text{ lb. per square inch}, \qquad f_s = 2640 \text{ lb. per square inch},$$
$$f_c = 98 \text{ lb. per square inch}$$

The unit bond stress u will also be very low. The building is therefore amply strong about the X-X axis.

10.10Y. Analysis about the Y-Y Axis.

From the theory of the moments of inertia of plane areas,

$$I_y = I_{y'} + Ad_y^2$$

where I_y = moment of inertia of a section about the Y-Y axis of the building.

$I_{y'}$ = moment of inertia of a section about its own y-y axis.

A = area of the section.

d_y = distance from the center of gravity of the section to the Y-Y axis of the building.

Let

$(EI)_{y_1}$ = EI of the average section taken above the floor.

I_{ys} = moment of inertia of the steel sections about the Y-Y axis.

Then

$$\begin{aligned} I_{ys} = {} & 4[1264 + 75.4(54 \times 12)^2] \\ & +6[2065 + 113.1(54 \times 12)^2] \\ & +4[2065 + 113.1(36 \times 12)^2] \\ & +4[2065 + 113.1(18 \times 12)^2] \\ & +2(2065) + 3(1790) \end{aligned}$$

$$+6[1790 + 100.6(36 \times 12)^2]$$
$$+6[1790 + 100.6(18 \times 12)^2]$$
$$= 6599 \times 10^5$$

and

$$E_s I_{ys} = 19{,}797 \times 10^{12}$$

Let

I_{y_1c} = moments of inertia of the reinforced-concrete sections about the Y-Y axis.

Then

$$\begin{aligned} I_{y_1c} = {} & 4[144{,}000 + 480(50.5 \times 12)^2] \\ & + 4[144{,}000 + 480(39.5 \times 12)^2] \\ & + 4[144{,}000 + 480(32.5 \times 12)^2] \\ & + 4[144{,}000 + 480(21.5 \times 12)^2] \\ & + 4[144{,}000 + 480(14.5 \times 12)^2] \\ & + 4[144{,}000 + 480(3.5 \times 12)^2] \\ & + 16[2560 + 480(54 \times 12)^2] \\ = {} & 48{,}514 \times 10^5 \end{aligned}$$

and

$$E_c I_{y_1c} = 9703 \times 10^{12}$$

But,

$$(EI)_{y_1} = E_s I_{ys} + E_c I_{y_1c}$$

Therefore,

$$(EI)_{y_1} = 295 \times 10^{14}$$

Let

$(EI)_{y_2}$ = EI of the average section taken through the equivalent floor slab.

I_{y_2c} = moment of inertia of the equivalent floor slab section about the Y-Y axis.

Then

$$I_{y_2c} = \tfrac{1}{12} \times 72.67 \times 12(108.67 \times 12)^3 = 1611 \times 10^8.$$

and

$$E_c I_{y_2c} = 3222 \times 10^{14}.$$

But

$$(EI)_{y_2} = E_s I_{ys} + E_c I_{y_2c}$$

Therefore,

$$(EI)_{y_2} = 3420 \times 10^{14}$$

The average value of $(EI)_y$ will therefore be

$$(EI)_y = \frac{532.6(EI)_{y_1} + 31.4(EI)_{y_2} + 6E_cI_{y,c}}{570}$$

or

$$(EI)_y = 497 \times 10^{14}$$

10.11Y. Transverse Period about the Y-Y Axis.

The transverse period of vibration about the Y-Y axis is expressed by

$$T_y = c\sqrt{\frac{wL^4}{(EI)_y g}}$$

where T_y = free fundamental period, seconds.
c = 1.50.
w = 12,000 lb. per inch
$L^4 = 2189 \times 10^{12}$.
$(EI)_y = 497 \times 10^{14}$.
g = 386 in. per second per second.

Therefore,

$$T_y = 1.50\sqrt{\frac{12{,}000 \times 2189 \times 10^{12}}{497 \times 10^{14} \times 386}} = 1.76 \text{ sec.}$$

During a destructive earthquake of period $T_p = 1.50$ sec.,

$$\frac{T_y}{T_p} = \frac{1.76}{1.50} = 1.17$$

and

$$\left(\frac{T_y}{T_p}\right)^{1/2} = 1.08$$

By Par. 6.8, the ratio $T/T_1 = 5.8$ when $c = 1.50$. Therefore, the first harmonic T_1 about the Y-Y axis will be

$$T_1 = \frac{1.76}{5.8} = 0.30 \text{ sec.}$$

10.12Y. Transverse Deflections, Moments, and Shears during Forced Vibration about the Y-Y Axis.

Entering Figs. 9, 10, and 11, and interpolating roughly between the curves marked $T/T_p = 1.1$ and $T/T_p = 1.5$, the following values of Y_u, M_u, and V_u are obtained:

c	Y_u	M_u	V_u
0 0	1.00	−6.45	3.90
0.2	0.55	−4.85	4.25
0.4	−0.70	−2.20	4.20
0 6	−2 40	−1.65	3 65
0.8	−4 45	−0.50	2.20
1.0	−6 55	0	0

By Eqs. (114) and (138),

$$m'L = 1.88\left(\frac{T_y}{T_p}\right)^{1/2} = 1.88 \times 1.08 = 2.03$$

$$m' = \frac{2.03}{6840} = 2.97 \times 10^{-4}$$

$$m'^2 = 8.81 \times 10^{-8}$$

$$m'^3 = 26.2 \times 10^{-12}$$

and

$$(EI)_y m'^2 F = 497 \times 10^{14} \times 8.81 \times 10^{-8} \times 0.12 = 525 \times 10^6$$

$$(EI)_y m'^3 F = 497 \times 10^{14} \times 26.2 \times 10^{-12} \times 0.12 = 156 \times 10^3$$

Therefore by Eqs. (161), (162), and (163), the actual deflections, moments, and shears are

(1) c	(2) y, in.	(3) Y, in.	(4) M, in.-lb.	(5) V, lb.	(6) V', lb.	(7) M', in.-lb.
0.00	0.12	0.00	-3390×10^6	609×10^3	1218×10^3	694×10^5
0.20	0 07	0.05	-2550×10^6	664×10^3	1328×10^3	756×10^5
0.40	−0 08	0 20	-1155×10^6	655×10^3	1310×10^3	747×10^5
0.60	−0 29	0.41	-866×10^6	570×10^3	1140×10^3	650×10^5
0.80	−0 53	0 65	-263×10^6	344×10^3	688×10^3	392×10^5
1.00	−0 79	0.91	0	0	0	0

the nomenclature being the same as in Par. 10.7X.

Column 3 shows that the maximum amplitude is at the top of the building and equals 0.91 in., which is also within the allowable amplitude of $0.001L = 6.84$ in. Column 6

shows that the maximum shear occurs at approximately two-tenths the height and equals 1,328,000 lb., and, by Column 7, the maximum moment taken by any set of vertical beams is 75,600,000 in.-lb.

10.13*Y*. Distribution of the Seismic Shear and Moment.

Investigating the average section by the methods of Chap. 8,

Steel beams	n	A	$r_{y'}$	$r_{y'}^2$	$I_{y'}$	Z
sa	4	75 4	4 10	16 8	1264	10.
sb	16	113.1	4 27	18 2	2065	6
sc	15	100.6	4 22	17 8	1790	7

$$Z = \frac{(L_b^2 + 36r_{y'}^2)}{I_{y'}}, \qquad L_b^2 = 12,996$$

By Eq. (190),

$$c_1 = \frac{6.61}{10.76} = 0.62$$

By Eq. (191),

$$c_2 = \frac{7.62}{10.76} = 0.71$$

Assuming that $p = 0.007$, then, by Eqs. (62) and (63), $K' = 0.097$, and,

Reinforced-concrete beams	n	b	d	A	$I_{y'}$	IK'	$r_{y'}^2$	G
ca	24	8	58	464	129,800	12,600	280	1 834
cb	16	60	6	360	1,080	105	3	124 7

$$G = \frac{(L_b^2 + 36r_{y'}^2)}{I_{y'}K'}$$

By Eq. (192),

$$c_3 = \frac{3.67}{32.28} = 0.1137$$

By Eq. (193),

$$c_4 = \frac{249.4}{32.28} = 7.71$$

and, by Eq. (198),

$$P_{sa} = \frac{1{,}328{,}000 \text{ lb.}}{264.0} = 5020 \text{ lb.}$$

By Eq. (195),

$P_{sb} = 8110$ lb., $P_{sc} = 7080$ lb., $P_{ca} = 44{,}200$ lb., $P_{cb} = 650$ lb.

In a like manner, by Eq. (199),

$$M_{sa} = \frac{75{,}600{,}000 \text{ in.-lb.}}{264.0} = 286{,}000 \text{ in.-lb.}$$

and, by Eq. (200),

$M_{sb} = 461{,}000$ in.-lb., $M_{sc} = 403{,}000$ in.-lb., $M_{ca} = 2{,}520{,}000$ in.-lb., $M_{cb} = 37{,}100$ in.-lb.

10.14*Y*. Shearing, Tensional, and Compressive Stresses.

Investigating the beam most rigid in the plane of bending, namely, *ca*, it is found, by Eq. (202), assuming that $j = 7/8$ and $k = 3/8$, that

$v = 109$ lb. per square inch, $f_s = 15{,}300$ lb. per square inch, $f_c = 568$ lb. per square inch

The steel area required is $A_s = pbd = 3.25$ sq. in. Three 1-in. rods and one 1-in. bar have a sectional area of 3.26 sq. in. and will be used. For this reinforcement, $o = 13.43$ in. Therefore, $u = 65$ lb. per square inch.

Hooked, deformed bars will be used. The following unit stresses are permissible, under the U. S. Navy Code, for a 2000-lb. concrete:

$u = 100$ lb. per square inch, $f_s = 18{,}000$ lb. per square inch, $f_c = 700$ lb. per square inch, $v = 180$ lb. per square inch

The building is therefore amply strong about the *Y*-*Y* axis.

CHAPTER 11

ASEISMIC DESIGN OF A SHORT BUILDING

11.1. Problem.

An earthquake-proof warehouse is to be built on a rectangular lot whose dimensions limit the plan of the building to 75 ft. 4 in. by 111 ft. 4 in. There are no adjoining buildings. There will be 12 floors and a basement. The story height is 12 ft. 0 in., and the distance between the top of a floor slab and the bottom of the spandrel beam below is 2 ft. 6 in. It is estimated that the equivalent depth of the average theoretical floor slab is 12 in. over the gross area of the plan. The lineal extent of the theoretical floor slab is

$$(1 + 12)1.0 \text{ ft.} = 13 \text{ ft.}$$

Assuming that the foundation slab will be 6 ft. thick, the total height of the building, L, becomes 162 ft. The lineal extent of the vertical beams therefore equals,

$$162 \text{ ft.} - 13 \text{ ft.} - 6 \text{ ft.} = 143 \text{ ft.}$$

Since the ratios L/d are:

$$\frac{162}{108.7} = 1.5 \qquad \text{and} \qquad \frac{162}{72.7} = 2.2$$

this building, by Par. 9.10, is classed as short.

11.2. Seismological and Geological Investigation.

Investigation of the site yielded the following data:

a. The locality is subject to considerable seismic activity.

b. There are no rifts or active faults in the immediate neighborhood, the nearest fault being 20 miles distant.

c. The dominant period of the locality is 0.3 sec.

d. Four test borings disclosed a dry, springy clay-and-sand mixture.

e. No ground water was encountered.

11.3. Foundation Amplitude.

A 3-ft. layer of broken stone will be interposed between the foundation and the soil. No special steps will be taken, however, to enable the building to slide freely. It is decided that the end conditions will be expressed by a value of $c = 1.00$.

Assuming that the maximum surface amplitude of the soil, excluding the case of synchronism, is 0.50 in., then, by Fig. 3, the amplitude at the bottom of the broken stone is 0.49 in. And, by Par. 9.6, the amplitude transmitted to the foundation is $F = \frac{1}{2} \times 0.49$ in. $= 0.25$ in.

11.4. Investigation of the Structural Frame.

The frame consists of a steel skeleton with reinforced-concrete roof, floor, and foundation slabs. The frame was designed to carry the dead, live, and wind loads, under the requirements of the local building code and in accordance with the provisions of Chap. 9. A series of reinforced-concrete bracing walls were assumed and distributed symmetrically about the plan. Figure 15 shows the horizontal cross section above the average floor and an elevation on the average floor. It is assumed that Young's moduli have the following values:

$$E_s = 3 \times 10^7, \qquad E_c = 2 \times 10^6$$

The following table gives the coefficients of the vertical beams shown in Fig. 15.

Beam	Section	A	$I_{x'}$	$I_{y'}$	$r_{x'}$	$r_{y'}$
sa	16 B.H. at 143 lb.	42.0	1,610	639	6.19	3 90
sb	16 B.H. at 177 lb.	52.2	2,078	821	6.31	3.96
sc	16 B.H. at 160 lb.	47.1	1,840	729	6.25	3.93
ca	R.C., 60 in. × 8 in.	480.0	2,560	144,000	2.31	17.32
cb	R.C., 8 in. × 60 in.	480.0	144,000	2,560	17.32	2.31

11.5X. Analysis about the X-X Axis.

Again, as in Par. 10.5X,

$$I_x = I_{x'} + Ad_x^2$$

Let

$(EI)_{x_1}$ = EI of the average section taken above the floor.

I_{xs} = moment of inertia of the steel sections about the X-X axis.

Then

$$\begin{aligned} I_{xs} &= 4[1610 + 42.0(36 \times 12)^2] \\ &\quad +10[2078 + 52.2(36 \times 12)^2] \\ &\quad + 4[2078 + 52.2(18 \times 12)^2] + 2(2078) \\ &\quad +10[1840 + 47.1(18 \times 12)^2] + 5(1840) \\ &= 1609 \times 10^5 \end{aligned}$$

and

$$E_s I_{xs} = 4827 \times 10^{12}$$

Let

I_{x_1c} = moment of inertia of the reinforced-concrete sections about the X-X axis.

Then

$$\begin{aligned} I_{x_1c} &= 24[2560 + 480(36 \times 12)^2] \\ &\quad +4[144{,}000 + 480(32.5 \times 12)^2] \\ &\quad +4[144{,}000 + 480(21.5 \times 12)^2] \\ &\quad +4[144{,}000 + 480(14.5 \times 12)^2] \\ &\quad +4[144{,}000 + 480(3.5 \times 12)^2] \\ &= 26{,}438 \times 10^5 \end{aligned}$$

and

$$E_c I_{x_1c} = 5288 \times 10^{12}.$$

But,

$$(EI)_{x_1} = E_s I_{xs} + E_c I_{x_1c}$$

Therefore,

$$(EI)_{x_1} = 101 \times 10^{14}$$

Let

$(EI)_{x_2}$ = EI of the average section taken through the equivalent floor slab.

I_{x_2c} = moment of inertia of the equivalent floor slab section about the X-X axis.

Then

$$I_{x_2c} = \frac{1}{12} \times 108.67 \times 12(72.67 \times 12)^3$$
$$= 721 \times 10^8$$

and

$$E_cI_{x_2c} = 1442 \times 10^{14}$$

But

$$(EI)_{x_2} = E_sI_{xs} + E_cI_{x_2c}$$

Therefore,

$$(EI)_{x_2} = 1490 \times 10^{14}$$

The average value of $(EI)_x$ will therefore be:

$$(EI)_x = \frac{143(EI)_{x_1} + 13(EI)_{x_2} + 6E_cI_{x_2c}}{162}$$

$$(EI)_x = 262 \times 10^{14}$$

11.6X. Transverse Period about the X-X Axis.

The transverse period is given by

$$T_x = c\sqrt{\frac{wL^4}{(EI)_xg}}$$

where T_x = free fundamental period, seconds.
c = 1.00.
$L^4 = (162 \times 12)^4 = 1428 \times 10^{10}$.
w = weight concentration, pounds per inch of L. The dead weight plus an allowance of 200 lb. per square foot of floor area under actual conditions of usage, is estimated as 48,600,000 lb. Therefore, w = 25,000 lb. per inch.
$(EI)_x = 262 \times 10^{14}$.
g = 386 in. per second per second.

Therefore,

$$T_x = 1.00\sqrt{\frac{25{,}000 \times 1428 \times 10^{10}}{262 \times 10^{14} \times 386}} = 0.19 \text{ sec.}$$

Applying the correction formula for short buildings of Par. 9.10,

$$T_{xt} = \sqrt{T_x^2\left(1 + \frac{d^2}{4c_1L^2}\right)}$$

Assuming that $c_1 = 0.26$,

$$L^2 = 378 \times 10^4, \qquad d^2 = 76 \times 10^4$$

Therefore,

$$T_{xt} = 0.21 \text{ sec.}$$

where T_{xt} is the true fundamental free period about the X-X axis.

Assuming that $T_p = 1.00$ sec.,

$$\frac{T_{xt}}{T_p} = 0.21, \qquad \left(\frac{T_{xt}}{T_p}\right)^{\frac{1}{2}} = 0.46$$

11.7X. Transverse Deflections, Moments, and Shears during Forced Vibration about the X-X Axis.

During forced vibration, the end conditions are qualitatively *fixed-free*. Therefore, entering Figs. 9, 10, and 11, and interpolating for a ratio $T/T_p = 0.21$, the following values of Y_u, M_u, and V_u are obtained:

c	Y_u	M_u	V_u
0.0	1.00	0.46	−0.64
0.2	1.01	0.31	−0.53
0.4	1.04	0.18	−0.42
0.6	1.10	0.09	−0.29
0.8	1.16	0.03	−0.15
1.0	1.21	0.00	0.00

By Eqs. (114) and (138),

$$m'L = 1.88 \times 0.46 = 0.86$$

$$m' = \frac{0.86}{1944} = 4.4 \times 10^{-4}$$

$$m'^2 = 19.6 \times 10^{-8}$$

$$m'^3 = 86.6 \times 10^{-12}$$

and

$$(EI)_x m'^2 F = 262 \times 10^{14} \times 19.6 \times 10^{-8} \times 0.25 = 128 \times 10^7$$

$$(EI)_x m'^3 F = 262 \times 10^{14} \times 86.6 \times 10^{-12} \times 0.25 = 57 \times 10^4$$

Consequently, by Eqs. (161), (162), and (163), the actual deflections, moments, and shears are

(1) c	(2) y, in.	(3) Y, in.	(4) M, in.-lb.	(5) V, lb.	(6) V', lb.	(7) M', in.-lb.
0.0	0.25	0.00	589×10^6	-365×10^3	730×10^3	416×10^5
0.2	0.25	0.00	397×10^6	-302×10^3	604×10^3	344×10^5
0.4	0.26	0.01	231×10^6	-240×10^3	480×10^3	274×10^5
0.6	0 27	0.02	115×10^6	-165×10^3	330×10^3	188×10^5
0.8	0 28	0.03	38×10^6	$-\ 87 \times 10^3$	174×10^3	99×10^5
1.0	0 30	0.05	0	0	0	0

the nomenclature being the same as in Par. 10.7X.

The maximum Y is at the top of the building and is equal to 0.05 in. This is the moment deflection. The corresponding shear deflection is obtained by the method of Par. 6.10, as follows:

$$y_s = y_m\left(\frac{d^2}{4c_1L^2}\right)$$

where y_s = shear deflection
y_m = moment deflection, $Y = 0.05$ in.
$d^2 = 76 \times 10^4$ $c_1 = 0.26$ $L^2 = 378 \times 10^4$

Therefore,

$$y_s = 0.01 \text{ in.}$$

The total maximum deflection at the top of the building is $(y_m + y_s)$ or 0.06 in. The maximum shear occurs at the bottom and equals 730,000 lb. The maximum moment taken by any set of vertical beams is 41,600,000 in.-lb.

11.8X. Distribution of the Seismic Shear and Moment.

Investigating the average section shown in Fig. 15, by the methods of Chap. 8,

Steel beams	n	A	$r_{x'}$	$r_{x'}^2$	$I_{x'}$	Z
sa	4	42.0	6.19	38 3	1610	8.93
sb	16	52.2	6.31	39.7	2078	6.96
sc	15	47.1	6.25	39.1	1840	7.83

$$Z = \frac{(L_b^2 + 36r_{x'}^2)}{I_{x'}}, \qquad L_b = 9.5 \text{ ft.} = 114 \text{ in.},$$

$$L_b^2 = 12{,}996$$

By Eq. (190),

$$c_1 = \frac{6.96}{8.93} = 0.78$$

By Eq. (191),

$$c_2 = \frac{7.83}{8.93} = 0.88$$

Assuming that $p = 0.007$, then, by Eqs. (62) and (63), $K' = 0.097$, and,

Reinforced concrete beams	n	b	d	A	$I_{x'}$	IK'	$r_{x'}^2$	G
ca	24	60	6	360	1,080	105	3	124 7
cb	16	8	58	464	129,800	12,600	280	1 834

$$G = \frac{(L_b^2 + 36r_{x'}^2)}{IK'}$$

By Eq. (192),

$$c_3 = \frac{249.4}{26.79} = 9.31$$

By Eq. (193),

$$c_4 = \frac{3.67}{26.79} = 0.137$$

Therefore, by Eq. (198),

$$P_{sa} = \frac{730{,}000 \text{ lb.}}{160.9} = 4540 \text{ lb.}$$

and, by Eq. (195),

$$P_{sb} = 5700 \text{ lb.}, \qquad P_{sc} = 5050 \text{ lb.}, \qquad P_{ca} = 480 \text{ lb.},$$
$$P_{cb} = 32{,}400 \text{ lb.}$$

Similarly, by Eq. (199),

$$M_{sa} = \frac{41{,}600{,}000 \text{ in.-lb.}}{160.9} = 258{,}000 \text{ in.-lb.}$$

and, by Eq. (200),

$$M_{sb} = 331{,}000 \text{ in.-lb.}, \qquad M_{sc} = 293{,}000 \text{ in.-lb.},$$
$$M_{ca} = 28{,}000 \text{ in.-lb.}, \qquad M_{cb} = 1{,}882{,}000 \text{ in.-lb.}$$

11.9*X*. Shearing, Tensional, and Compressive Stresses.

Investigating the beam most rigid in the plane of bending, namely, *cb*, it is found, by Eq. (202), assuming that $j = \frac{7}{8}$ and $k = \frac{3}{8}$, that

$v = 80$ lb. per square inch, $f_s = 11{,}400$ lb. per square inch, $f_c = 427$ lb. per square inch

The building is therefore amply strong about the *X-X* axis.

11.10*Y*. Analysis about the *Y-Y* Axis.

Referring to Par. 10.10*Y*,

$$I_y = I_{y'} + Ad_y^2$$

Let

$(EI)_{y_1}$ = *EI* of the average section taken above the floor.

I_{ys} = moment of inertia of the steel sections about the *Y-Y* axis.

Then,

$$\begin{aligned} I_{ys} = {} & 4[639 + 42.0(54 \times 12)^2] \\ & +6[821 + 52.2(54 \times 12)^2] \\ & +4[821 + 52.2(36 \times 12)^2] \\ & +4[821 + 52.2(18 \times 12)^2] \\ & +2(821) + 3(729) + 6[729 + 47.1(36 \times 12)^2] \\ & +6[729 + 47.1(18 \times 12)^2] \\ = {} & 3173 \times 10^5. \end{aligned}$$

and

$$E_sI_{ys} = 9519 \times 10^{12}$$

Let,

I_{yc} = moments of inertia of the reinforced-concrete sections about the *Y-Y* axis.

Then,

$$\begin{aligned} I_{yc} = {} & 4[144{,}000 + 480(50.5 \times 12)^2] \\ & +4[144{,}000 + 480(39.5 \times 12)^2] \\ & +4[144{,}000 + 480(32.5 \times 12)^2] \\ & +4[144{,}000 + 480(21.5 \times 12)^2] \\ & +4[144{,}000 + 480(14.5 \times 12)^2] \\ & +4[144{,}000 + 480(3.5 \times 12)^2] \\ & +16[2560 + 480(54 \times 12)^2] \\ = {} & 48{,}514 \times 10^5 \end{aligned}$$

and

$$E_c I_{y_1c} = 9703 \times 10^{12}$$

But

$$(EI)_{y_1} = E_s I_{ys} + E_c I_{y_1c}$$

Therefore,

$$(EI)_{y_1} = 192 \times 10^{14}$$

Let

$(EI)_{y_2}$ = EI of the average section taken through the equivalent floor slab.

I_{y_2c} = moment of inertia of the equivalent floor slab section about the Y-Y axis.

Then,

$$I_{y_2c} = \tfrac{1}{12} \times 72.67 \times 12(108.67 \times 12)^3 = 1611 \times 10^8$$

and

$$E_c I_{y_2c} = 3222 \times 10^{14}$$

But

$$(EI)_{y_2} = E_s I_{ys} + E_c I_{y_2c} = 3317 \times 10^{14}$$

The average value of $(EI)_y$ will therefore be

$$(EI)_y = \frac{143(EI)_{y_1} + 13(EI)_{y_2} + 6E_c I_{y_2c}}{162}$$

or

$$(EI)_y = 556 \times 10^{14}$$

11.11*Y*. Transverse Period about the *Y*-*Y* Axis.

Again,

$$T_y = c\sqrt{\frac{wL^4}{(EI)_y g}}$$

$c = 1.00, \quad w = 25{,}000, \quad L^4 = 1428 \times 10^{10},$
$(EI)_y = 556 \times 10^{14}, \quad g = 386$

Therefore,

$$T_y = 0.13 \text{ sec.}$$

Applying the correction formula for short buildings, of Par. 9.10,

$$T_{yt} = \sqrt{T_y^2\left(1 + \frac{d^2}{4c_1L^2}\right)} .$$

$c_1 = 0.26, \qquad L^2 = 378 \times 10^4,$

$$d^2 = (108.67 \times 12)^2 = 17 \times 10^5$$

Therefore,

$$T_{yt} = 0.16 \text{ sec.}$$

where T_{yt} is the true fundamental free period about the Y-Y axis.

Assuming that $T_p = 1.00$ sec.,

$$\frac{T_{yt}}{T_p} = 0.16, \qquad \left(\frac{T_{yt}}{T_p}\right)^{1/2} = 0.40$$

11.12Y. Transverse Deflections, Moments, and Shears during Forced Vibration about the Y-Y Axis.

Entering Figs. 9, 10, and 11, and interpolating for a ratio $T/T_p = 0.16$, the following values of Y_u, M_u, and V_u were obtained:

c	Y_u	M_u	V_u
0 0	1 00	0 35	−0 48
0.2	1.01	0.23	−0 40
0.4	1.04	0 14	−0 32
0.6	1.08	0 07	−0.22
0.8	1.12	0.02	−0 11
1.0	1.16	0	0

By Eqs. (114) and (138),

$$m'L = 1.88 \times 0.40 = 0.75$$

$$m' = \frac{0.75}{1944} = 3.9 \times 10^{-4}$$

$$m'^2 = 15.2 \times 10^{-8}$$

$$m'^3 = 59.3 \times 10^{-12}$$

and

$$(EI)_y m'^2 F = 556 \times 10^{14} \times 15.2 \times 10^{-8} \times 0.25 = 212 \times 10^7$$

$$(EI)_y m'^3 F = 556 \times 10^{14} \times 59.3 \times 10^{-12} \times 0.25 = 825 \times 10^3$$

Consequently, by Eqs. (161), (162), and (163), the actual deflections, moments, and shears are

(1) c	(2) y, in.	(3) Y, in.	(4) M, in.-lb.	(5) V, lb.	(6) V', lb.	(7) M', in.-lb.
0.0	0.25	0.00	743×10^6	-396×10^3	792×10^3	452×10^5
0.2	0.25	0.00	488×10^6	-330×10^3	660×10^3	376×10^5
0.4	0.26	0.01	297×10^6	-264×10^3	528×10^3	301×10^5
0.6	0.27	0.02	148×10^6	-182×10^3	364×10^3	208×10^5
0.8	0.28	0.03	42×10^6	-91×10^3	182×10^3	104×10^5
1.0	0.29	0.04	0	0	0	0

the nomenclature being the same as in Par. 10.7X.

The maximum Y is equal to 0.04 in. This is the moment deflection. The corresponding shear deflection is obtained as in Par. 11.7X, or

$$y_s = y_m\left(\frac{d^2}{4c_1L^2}\right)$$

where y_s = shear deflection
y_m = moment deflection, $Y = 0.04$ in.
$d^2 = 17 \times 10^5$ $c_1 = 0.26$ $L^2 = 378 \times 10^4$

Therefore,

$$y_s = 0.02 \text{ in.}$$

The total maximum deflection is $(y_m + y_s)$ or 0.06 in. The maximum shear occurs at the bottom and equals 792,000 lb. The maximum moment taken by any set of vertical beams is 45,200,000 in.-lb.

11.13Y. Distribution of the Seismic Shear and Moment.

Investigating the average section, by the methods of Chap. 8,

Steel beams	n	A	$r_{y'}$	$r_{y'}^2$	$I_{y'}$	Z
sa	4	42.0	3.90	15.2	639	21.2
sb	16	52.2	3.96	15.7	821	16.5
sc	15	47.1	3.93	15.5	729	18.6

$$Z = \frac{(L_b^2 + 36r_{y'}^2)}{I_{y'}}, \qquad L_b^2 = 12{,}996$$

By Eq. (190),

$$c_1 = \frac{16.5}{21.2} \doteq 0.78$$

By Eq. (191),

$$c_2 = \frac{18.6}{21.2} = 0.88$$

Assuming that $p = 0.007$, then, by Eqs. (62) and (63), $K' = 0.097$, and,

Reinforced-concrete beams	n	b	d	A	$I_{y'}$	IK'	$r_{y'}^2$	G
ca	24	8	58	464	129,800	12,600	280	1.834
cb	16	60	6	360	1,080	105	3	124.7

$$G = \frac{(L_b^2 + 36r_{y'}^2)}{IK'}$$

By Eq. (192),

$$c_3 = \frac{3.67}{63.6} = 0.0577$$

By Eq. (193),

$$c_4 = \frac{249.4}{63.6} = 3.92$$

and, by Eqs. (195) and (198),

$$P_{sa} = \frac{792{,}000 \text{ lb.}}{461.6} = 1720 \text{ lb.,} \qquad P_{sb} = 2200 \text{ lb.,}$$

$$P_{sc} = 1950 \text{ lb.,} \qquad P_{ca} = 29{,}800 \text{ lb.,} \qquad P_{cb} = 440 \text{ lb.}$$

Similarly, by Eqs. (199) and (200),

$$M_{sa} = \frac{45{,}200{,}000 \text{ in.-lb.}}{461.6} = 98{,}000 \text{ in.-lb.,} \quad M_{sb} = 126{,}000$$

in.-lb., $M_{sc} = 111{,}000$ in.-lb., $M_{ca} = 1{,}697{,}000$ in.-lb., and $M_{cb} = 25{,}000$ in.-lb.

11.14*Y*. Shearing, Tensional, and Compressive Stresses.

Investigating the beam most rigid in the plane of bending, namely, *ca*, it is found, by Eq. (202), assuming that $j = 7/8$, and $k = 3/8$, that

$v = 73$ lb. per square inch, $f_s = 10{,}300$ lb. per square inch, $f_c = 385$ lb. per square inch

The building is therefore amply strong about the *Y*-*Y* axis.

INDEX

A

S

T

U

V

W